Breakdown-point for Spatially and Temporally Correlated Observations

M.G. Genton[1]

Department of Statistics, North Carolina State University, USA
E-mail: genton@stat.ncsu.edu

Summary. In this paper, we implement a new definition of breakdown in both finite and asymptotic samples with correlated observations arising from spatial statistics and time series. In such situations, existing definitions typically fail because parameters can sometimes breakdown to zero, i.e. the center of the parameter space. The reason is that these definitions center around defining an explicit critical region for either the parameter or the objective function. If for a particular outlier constellation the critical region is entered, breakdown is said to occur. In contrast to the traditional approach, we use a definition that leaves the critical region implicit but still encompasses all previous definitions of breakdown in linear and nonlinear regression settings. We provide examples involving simultaneously specified spatial autoregressive models, as well as autoregressions from time series, for illustration. In particular, we show that in this context the least median of squares estimator has a breakdown-point much lower than the familiar 50%.

Key words: AR, Breakdown-point, Bias, Correlation, QAR, Robustness, SAR, Spatial statistics, Time series

1 Introduction

Data sets arising from environmental sciences are typically correlated in time, space, or space-time, and thus there is a need to construct new tools to measure robustness for correlated data. Existing tools to study the robustness of statistical procedures have been mainly developed for independent observations. The most familiar ones include the influence function (Hampel, 1971), the change-of-variance function (Rousseeuw, 1981), the max-bias curve (Hampel et al., 1986), and the breakdown-point (Hampel, 1971). Only few attempts have been made to extend these definitions to the setting of correlated observations. Künsch (1984) followed by Martin and Yohai (1986) have proposed two different definitions of the influence function for time series. Hössjer (1991) extended the change-of-variance function in order to study the effect of unexpected correlations in the data. Recently, Genton (2001) broadened the definition of the change-of-variance function to the spatial statistics context, in order to explore the effects of both dependencies and outliers on the variance of variogram estimators. In this paper, we use a new definition of breakdown-point in the context of correlated observations and provide simple illustrative examples arising from spatial statistics and time series.

R. Dutter et al. (eds.), *Developments in Robust Statistics*

The issue of qualitative robustness and especially the definition of breakdown has made considerable progress over the last three decades. Informally, breakdown of a statistical procedure means that the procedure no longer conveys useful information on the true underlying model. A natural way of defining location breakdown-point was proposed by Hodges (1967) and called *tolerance*. It consists in studying the behavior of the estimate when some of the observations are replaced with positive or negative extreme values. Hampel (1971) implemented breakdown in the setting of parameter estimation as the fraction of contamination (or outliers) that suffices to drive the estimator beyond all bounds, or to the edge of the parameter space if the latter is bounded. The breakdown-point has then been extended to finite samples (Donoho and Huber, 1983), dependent observations (Martin and De Jong, 1977; Martin, 1980), test statistics (He et al., 1990; He, 1991), and nonlinear regression models (Stromberg and Ruppert, 1992; Sakata and White, 1995, 1998). Especially, Stromberg and Ruppert (1992) and Sakata and White (1995) convincingly argue that the bias in the parameter estimates is not always a good criterion to assess breakdown of an estimator. Instead, Stromberg and Ruppert (1992) propose to consider the fraction of contamination that drives at least one of the fitted values to its supremum or infimum. Sakata and White (1995, 1998) argue that the fitted value may sometimes not be a satisfactory criterion either, and therefore propose several alternative criterion functions to assess breakdown. Genton (1998), as well as Ma and Genton (2000), define respectively a spatial and a temporal breakdown-point. However, both definitions are local in the sense that they describe breakdown only for a fixed lag vector in space or time.

The remainder of the paper is set up as follows. In Section 2, we provide evidence that the above definitions of breakdown-point can become unreliable in situations arising from spatial statistics and time series. Section 3 describes a definition of breakdown-point that is suitable for the context of correlated observations. We provide further examples of breakdown in the asymptotic case in Section 4, and we conclude in Section 5.

2 Examples from Spatial Statistics and Time Series

Although the alternative definitions of breakdown-point described above cover a wide range of models and estimators, one can easily construct examples that are not covered by those definitions. A very simple example involving a vector $\mathbf{y} = (Y_1, \ldots, Y_n)^T$ of n correlated observations on a 2-dimensional lattice is given by the simultaneously specified spatial autoregressive (SAR) model

$$\mathbf{y} = \rho W \mathbf{y} + \boldsymbol{\epsilon}, \tag{1}$$

with bounded parameter space for $\rho \in (-1, 1)$, and the maximum likelihood (ML) estimator of ρ. Such models have important applications, for instance in the context of pollution analysis, socio-economic studies, and agricultural trials. Here W is an $n \times n$ matrix of nonnegative weights describing the degree of interaction between

neighbor locations in the plane, and such that $W\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is a vector of ones of dimension n. Figure 1 depicts two neighborhood structures: the left panel is a full nearest neighbor structure, whereas the right panel is a quadrant nearest neighbor structure. Models with the latter structure are often referred to as quadrant autoregressions (QAR). The white discs indicate the neighbors of the black one. The vector

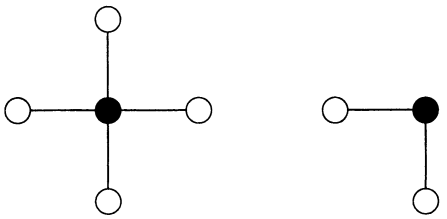

Fig. 1. Nearest neighbor structure: full (left panel) and quadrant (right panel). The white discs indicate the neighbors of the black one.

$\boldsymbol{\epsilon}$ is assumed to have i.i.d. N(0,1) components and the vector $\mathbf{y}$ is multivariate normal $N_n(\mathbf{0}, \Sigma)$, where the covariance matrix is equal to $\Sigma = [(I - \rho W^T)(I - \rho W)]^{-1}$. Note that the vector $\boldsymbol{\epsilon}$ is correlated with $\mathbf{y}$ since $\mathrm{Cov}(\boldsymbol{\epsilon}, \mathbf{y}) = (I - \rho W^T)^{-1}$. Thus, the least squares estimator of ρ is not consistent (Whittle, 1954). However, the maximum likelihood estimator $\hat{\rho}$ of ρ is consistent and is the solution of the following equation (Ord, 1975)

$$\begin{aligned}\mathrm{tr}[W(I - \rho W)^{-1}](\mathbf{y}^T\mathbf{y} - 2\rho\mathbf{y}^T W\mathbf{y} + \rho^2\mathbf{y}^T W^T W\mathbf{y}) \\ +n(\rho\mathbf{y}^T W^T W\mathbf{y} - \mathbf{y}^T W\mathbf{y}) = 0. \qquad (2)\end{aligned}$$

Suppose $\mathbf{y}$ is observed with error as $\tilde{\mathbf{y}} = \mathbf{y} + \zeta\mathbf{e}_i$, where $\mathbf{e}_i$ has a nonzero component only at index i, where it is 1. How will the ML estimator of ρ based on the contaminated sample $\tilde{\mathbf{y}}$ be affected?

First, let us consider the case of a quadrant neighbor structure. If we label the observations on the lattice from South to North and West to East, then the matrix W is lower triangular. Because the diagonal of W is always zero (observations are not neighbors of themselves), the eigenvalues of W are all equal to zero for the QAR model. Thus, the left factor of the first term in (2) is zero and we obtain an explicit solution for the maximum likelihood estimator of ρ given by

$$\hat{\rho} = \frac{\mathbf{y}^T W\mathbf{y}}{\mathbf{y}^T W^T W\mathbf{y}}. \qquad (3)$$

Based on the contaminated sample $\tilde{\mathbf{y}}$, we get

$$\hat{\rho} = \frac{\mathbf{y}^T W\mathbf{y} + \zeta(\mathbf{e}_i^T W\mathbf{y} + \mathbf{y}^T W\mathbf{e}_i) + \zeta^2\mathbf{e}_i^T W\mathbf{e}_i}{\mathbf{y}^T W^T W\mathbf{y} + \zeta(\mathbf{e}_i^T W^T W\mathbf{y} + \mathbf{y}^T W^T W\mathbf{e}_i) + \zeta^2\mathbf{e}_i^T W^T W\mathbf{e}_i}. \qquad (4)$$

Note that $\mathbf{e}_i^T W\mathbf{e}_i = 0, \forall i$, whereas $\mathbf{e}_i^T W^T W\mathbf{e}_i \neq 0$ (except when W has a column of zeros, i.e. when the outlier is located at the North-East corner of the lattice).

Thus, as $\zeta \to \infty$, we have $\hat{\rho} \to 0$ whatever the true value of ρ, i.e., whatever the realization of the uncontaminated sample $\mathbf{y}$. The ML estimator no longer conveys any useful information on ρ and the estimate is totally dictated by the contamination. In this simple quadrant autoregressive spatial model, the ML estimator breaks down to $\hat{\rho} = 0$ with one extreme outlier. Note that $\rho = 0$ is at the center of the parameter space.

Let us now consider the full nearest neighbor structure, in which case an explicit form for ρ does not exist. However, if we suppose that the vector $\mathbf{y}$ in (2) is observed with error as $\tilde{\mathbf{y}} = \mathbf{y} + \zeta \mathbf{e}_i$, the solution ρ must satisfy the following equation

$$\zeta^2 h_2(\rho) + \zeta h_1(\rho) + h_0(\rho) = 0, \tag{5}$$

where $h_0(\rho)$ is defined by the left-hand side of (2), and

$$\begin{aligned} h_1(\rho) &= k(\rho)\big(\mathbf{e}_i^T\mathbf{y} - 2\rho\mathbf{e}_i^T W\mathbf{y} + \rho^2\mathbf{e}_i^T W^T W\mathbf{y}\big) + n\big(\rho\mathbf{e}_i^T W^T W\mathbf{y} - \mathbf{e}_i^T W\mathbf{y}\big), \\ h_2(\rho) &= k(\rho)\big(1 + \rho^2 a\big) + n\rho. \end{aligned} \tag{6}$$

Here $k(\rho) = \mathrm{tr}[W(I - \rho W)^{-1}]$ and $a = \mathbf{e}_i^T W^T W\mathbf{e}_i$. As $\zeta \to \infty$, the expression $h_2(\rho)$ must tend to zero in order for (5) to hold. Denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the matrix W. Because $h_2(0) = 0$ and

$$\begin{aligned} h_2'(\rho) &= (1 + \rho^2 a)\sum_{j=1}^{n}\Big(\frac{1}{\lambda_j} - \rho\Big)^{-2} + 2\rho a\sum_{j=1}^{n}\Big(\frac{1}{\lambda_j} - \rho\Big)^{-1} + na \\ &\geq \sum_{j=1}^{n}\Big[\rho\Big(\frac{1}{\lambda_j} - \rho\Big)^{-1} + a\Big]^2, \\ &\geq 0, \end{aligned} \tag{7}$$

i.e. $h_2(\rho)$ is monotone on the interval $(-1, 1)$, the estimator $\hat{\rho} \to 0$. Here again, as $\zeta \to \infty$, the ML estimator of ρ breaks down to zero and no longer conveys any useful information on ρ whatever the realization of the uncontaminated sample $\mathbf{y}$. This situation can not be accommodated by the classical definitions of breakdown mentioned previously.

It is interesting to note that the autoregressive time series model of order one, AR(1), is a particular QAR model. Effectively, the matrix W describes neighbors only on the left (i.e. depending on the past) in the time series case. This yields a matrix W full of zero entries, except that $W_{ij} = 1$ when $j - i = 1$. Thus, the matrix $W^T W = I$, and the ML estimator of the parameter ρ of the AR(1) model $Y_i = \rho Y_{i-1} + \epsilon_i$ takes the usual expression

$$\hat{\rho} = \frac{\mathbf{y}^T W\mathbf{y}}{\mathbf{y}^T W^T W\mathbf{y}} = \frac{\sum_{i=2}^{n} Y_i Y_{i-1}}{\sum_{i=2}^{n} Y_i^2}. \tag{8}$$

Consequently, we observe again breakdown to zero when the vector $\mathbf{y}$ is observed with one extreme outlier and $\zeta \to \infty$.

The form of breakdown discussed above for the SAR, QAR, and AR(1) model typically rules out the classical definition of Hampel, because the estimator does not diverge, but instead tends to zero. It also violates the definition of breakdown based on supremum bias which is reached when $\hat{\rho}$ tends to $+1$ for negative ρ, or when $\hat{\rho}$ tends to -1 for positive ρ. Finally, the alternative definition of breakdown proposed by Stromberg and Ruppert (1992), as well as Sakata and White (1995), fails too. It is based on some criterion function such as the model fit, reaching its supremum or infimum. Indeed, this would imply breakdown to $+1$ or -1, but not to 0, the center of the parameter space.

3 Breakdown-point for Correlated Observations

Given the drawbacks of the previous definitions available, Genton and Lucas (2000) recently proposed a new definition of breakdown in finite samples with an extension to asymptotic breakdown. Previous definitions center around defining an explicit critical region for either the parameter or the objective function. If for a particular outlier constellation the critical region is entered, breakdown is said to occur. In contrast to the traditional approach, the definition of the critical region is now implicit. This allows to cover the previous definitions, as well as situations with correlated observations as demonstrated below.

3.1 Finite Sample Breakdown-point

Consider a vector $\mathbf{y} = (Y_1, \ldots, Y_n)^T$ of n i.i.d. or correlated observations and denote by $\hat{\boldsymbol{\theta}}(\mathbf{y})$ an estimator of a multidimensional parameter $\boldsymbol{\theta}$ based on the vector of observations $\mathbf{y}$. We introduce outliers trough a contaminating vector $\mathbf{z}_m^\zeta$, where m represents the number of outliers and ζ their magnitude. In the examples of Section 2, we considered $\mathbf{z}_1^\zeta = \zeta e_i$, but other outliers patterns can be studied, such as additive, replacement, or innovation outliers. The set of allowable outlier constellations is represented by $\mathcal{Z}_m^\zeta$. We define a measure of badness, $R_n(\boldsymbol{\theta}, \mathbf{y}) \in \mathbb{R}^+$, which could represent bias or model fit for instance. We further define a badness set by

$$R_n^*(\mathbf{y}, \mathbf{z}_m^\zeta, \mathcal{Y}_n) = \bigcup_{\mathbf{y}' \in \mathcal{Y}_n} \left\{ R_n(\hat{\boldsymbol{\theta}}(\mathbf{y}' + \mathbf{z}_m^\zeta), \mathbf{y}) \right\}, \tag{9}$$

where $\mathcal{Y}_n$ is a neighborhood of the uncontaminated sample $\mathbf{y}$, representing the set of allowable alternative uncontaminated samples that might have been realized. Genton and Lucas (2000) define the breakdown-point ε_n of the estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ under the badness measure R_n at the uncontaminated sample $\mathbf{y}$ for the set of allowable outlier constellations $\mathcal{Z}_m^\zeta$ and alternative uncontaminated samples $\mathcal{Y}_n$ by

$$\varepsilon_n \equiv \min \left\{ \frac{m-1}{n} \,\middle|\, \exists\, \mathbf{z}_m^\zeta \in \mathcal{Z}_m^\zeta \text{ such that } R_n^*(\mathbf{y}, \mathbf{z}_m^\zeta, \mathcal{Y}_n) \text{ is a finite set} \right\}. \tag{10}$$

This definition looks for the smallest fraction of outliers for which the set of possible badness values can take on only a finite number of values. Note that in most cases, breakdown is concerned with extreme outliers such that we will focus on the case $\zeta \to \infty$. If the sample space is continuous, this signals that most, if not all of the information on the uncontaminated sample has been lost.

For example, consider the sample mean $\hat{\theta}$ with one ($m = 1$) extreme outlier at $+\infty$ and $\mathcal{Y}_n = \mathbb{R}^n$. Denote the outlier by Z_1^∞. If badness is defined as bias, $R_n(\theta, \mathbf{y}) = |\theta - \hat{\theta}(\mathbf{y})|$, then $R_n^*(\mathbf{y}, Z_1^\infty, \mathbb{R}^n) = \{\infty\}$, which is a finite set. Therefore, the breakdown-point of the mean is given by $\varepsilon_n = (1-1)/n = 0$.

Next, consider the examples arising from spatial statistics and time series introduced earlier with one additive outlier, see Section 2. It was shown that the ML estimator in this case tends to zero for one extreme outlier ($\zeta \to \infty$), independently of the sample. Using the definition proposed by Genton and Lucas (2000) and badness equal to bias, the ML breakdown-point in this setting equals zero as we have $R_n^*(\mathbf{y}, \mathbf{z}_1^\infty, \mathbb{R}^n) = \{|0 - \hat{\theta}(\mathbf{y})|\}$, where $\mathbf{z}_1^\zeta = \zeta e_i$ and $\hat{\theta}(\mathbf{y}) = \hat{\rho}$. In contrast, consider the outlier configuration $\mathbf{z}_m^\zeta = (0, \ldots, 0, \zeta, \zeta, \ldots, \zeta, 0, \ldots, 0)$, where the patch of ζs has length m. If badness is bias, and for the AR(1) model, it is easy to check that this results in a badness of $|(m-1)/m - \hat{\theta}(\mathbf{y})|$ for ζ diverging to infinity. Similarly, if we have an alternating outlier patch $\zeta, -\zeta, \zeta, \ldots$ of length m, the badness is $|-(m-1)/m - \hat{\theta}(\mathbf{y})|$. Supremum badness is reached if $m = n$, such that the breakdown-point following Sakata-White in this case is close to one. This seems inappropriate, as the ML estimator has already lost all its informational content under one extreme outlier. The definition introduced by Genton and Lucas (2000) automatically detects this information loss and assigns a breakdown-point of zero to the ML estimator.

3.2 Asymptotic Breakdown-point

The definition of breakdown-point is extended to the asymptotic case by introducing the stochastic processes $\mathbf{y} = \{Y_i\}_{i=1}^\infty$ and $\mathbf{z}_\pi^\zeta = \{Z_i^\zeta\}_{i=1}^\infty$. The interpretation is similar to that in the previous section. The subscript π for the contaminating process indicates the fraction or probability of contamination, for example $P(Z_i^\zeta = \zeta) = 1 - P(Z_i^\zeta = 0) = \pi$ for all i. The badness measure is now a functional $R(\boldsymbol{\theta}, \mathbf{y}) \in \mathbb{R}^+$, and the badness set is defined by

$$R^*(\mathbf{y}, \mathbf{z}_\pi^\zeta, \mathcal{Y}) = \bigcup_{\mathbf{y}' \in \mathcal{Y}} \left\{ R(\hat{\boldsymbol{\theta}}(\mathbf{y}' + \mathbf{z}_\pi^\zeta), \mathbf{y}) \right\}, \tag{11}$$

with $\mathcal{Y}$ the set of alternative uncontaminated processes allowed. Genton and Lucas (2000) define the asymptotic breakdown-point ε of the estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ under badness measure R at the uncontaminated process $\mathbf{y}$ for the set of allowable contaminating processes $\mathcal{Z}_\pi^\zeta$ and alternative uncontaminated processes $\mathcal{Y}$ by

$$\varepsilon \equiv \inf \left\{ \pi \middle| \exists\, \mathbf{z}_\pi^\zeta \in \mathcal{Z}_\pi^\zeta \text{ such that } R^*(\mathbf{y}, \mathbf{z}_\pi^\zeta, \mathcal{Y}) \text{ is a finite set} \right\}. \tag{12}$$

Here again, the definition looks for the smallest fraction of outliers for which the set of possible badness values can take on only a finite number of values. The key assumption is that the parameter space is not discrete, in which case the estimator can no longer clearly distinguish between alternate uncontaminated samples.

To illustrate the definition, consider the mean as a location estimator in the i.i.d. setting,

$$\hat{\theta}(\mathbf{y}) = \int Y_i P(dY_i). \tag{13}$$

As a badness measure, consider the bias $|\hat{\theta} - \theta|$. This should give us the Hampel breakdown-point of the mean. Let $\mathcal{Y}$ denote the set of processes with finite mean. For the i.i.d. contaminating process $P(Z_i^\zeta = \zeta) = 1 - P(Z_i^\zeta = 0) = \pi$ with $\zeta \to \infty$, it is then easy to show that for every positive π arbitrarily close to 0, the badness set R^* equals $\{\infty\}$, which is a finite set. Consequently, the breakdown-point of the mean is zero. A similar line of argument can be used to show that the median has a breakdown-point of 50%.

4 An Example Involving Asymptotic Breakdown-point

In this section, we present the computation of the asymptotic breakdown-point of the least median of squares (LMS, see Rousseeuw, 1984) applied to a unidimensional simultaneously specified spatial autoregressive model of order 1. The SAR model is defined by

$$Y_i = \frac{\rho}{2}(Y_{i-1} + Y_{i+1}) + \epsilon_i, \tag{14}$$

where the innovations ϵ_i form a Gaussian i.i.d. process and the parameter is such that $|\rho| < 1$ to insure stationarity. The assumption of Gaussianity is not strictly necessary, but serves our expositional purposes. We assume that badness is measured by bias, $|\hat{\rho}(\mathbf{y}' \dot{+} \mathbf{z}^\zeta) - \hat{\rho}(\mathbf{y})|$ or $|\hat{\rho}(\mathbf{y}' \dot{+} \mathbf{z}^\zeta) - \rho|$ for $\mathbf{y}' \in \mathcal{Y}$, where $\mathcal{Y}$ is the set of strictly stationary unidimensional Gaussian SAR processes. Note that we can characterize the elements $\mathbf{y}'$ of $\mathcal{Y}$ by the value of the autoregressive parameter ρ'.

Consider the i.i.d. additive outlier processes

$$P(Z_i^\zeta = \zeta) = P(Z_i^\zeta = -\zeta) = (1 - P(Z_i^\zeta = 0))/2 = \pi/2.$$

The LMS estimator $\hat{\rho}_{LMS}$ of ρ minimizes

$$\text{median}\left(\epsilon_i + Z_i^\zeta - \frac{\hat{\rho}_{LMS}}{2}(Z_{i-1}^\zeta + Z_{i+1}^\zeta) + \frac{\rho - \hat{\rho}_{LMS}}{2}(Y_{i-1} + Y_{i+1})\right)^2, \tag{15}$$

which means that it solves

$$0.5 = P\left(\left(\epsilon_i + Z_i^\zeta - \frac{\hat{\rho}_{LMS}}{2}(Z_{i-1}^\zeta + Z_{i+1}^\zeta) + \frac{\rho - \hat{\rho}_{LMS}}{2}(Y_{i-1} + Y_{i+1})\right)^2 < c\right) \tag{16}$$

with respect to c. The righthandside of (16) can be written as a sum of $3^3 = 27$ terms depending on Z_i^ζ, Z_{i-1}^ζ, and Z_{i+1}^ζ, each taking the 3 possible values $+\zeta$, $-\zeta$,

or 0. If we let $\zeta \to \infty$, we see that there are only 4 interesting values of $\hat{\rho}_{LMS}$, namely $\hat{\rho}_{LMS}$ equal to ρ, $+1$, -1, or 0. We can now rewrite (16) as

$$c = \begin{cases} P^{-1}_{(\epsilon_i + \frac{\rho}{2}(Y_{i-1}+Y_{i+1}))^2}(0.5(1-\pi)^{-1}) & \text{for } \hat{\rho}_{LMS} = 0, \\ P^{-1}_{\epsilon_i^2}((1-\pi)^{-1}(2-4\pi+3\pi^2)^{-1}) & \text{for } \hat{\rho}_{LMS} = \rho, \\ P^{-1}_{(\epsilon_i + \frac{\rho-1}{2}(Y_{i-1}+Y_{i+1}))^2}(2(4-12\pi+14\pi^2-5\pi^3)^{-1}) & \text{for } \hat{\rho}_{LMS} = +1, \\ P^{-1}_{(\epsilon_i + \frac{\rho+1}{2}(Y_{i-1}+Y_{i+1}))^2}(2(4-12\pi+14\pi^2-5\pi^3)^{-1}) & \text{for } \hat{\rho}_{LMS} = -1. \end{cases} \tag{17}$$

where $P_X^{-1}(\cdot)$ is the inverse c.d.f. corresponding to the random variable X. It is clear from these formulas that for π sufficiently small, $\hat{\rho}_{LMS} = \rho$ gives the lower value of c. For ρ near 0 or ± 1 and π sufficiently large, however, one of the other branches may dominate. Assume $\mathcal{Y}$ is the set of stationary unidimensional Gaussian SAR processes characterized by $|\rho| < 1$. In order to compute (17), we consider

$$\mathrm{E}\left(\epsilon_i + \frac{\rho}{2}(Y_{i-1} + Y_{i+1})\right) = 0 \tag{18}$$

$$\begin{aligned} \mathrm{Var}(\epsilon_i + \frac{\rho}{2}(Y_{i-1} + Y_{i+1})) &= 1 + \frac{\rho^2}{2}\mathrm{Var}(Y_{i-1}) + \frac{\rho^2}{2}\mathrm{Cov}(Y_{i-1}, Y_{i+1}) \\ &\quad + 2\rho\mathrm{Cov}(\epsilon_i, Y_{i-1}) \\ &= \frac{\rho^4 - \rho^2 + 2}{2(1-\rho^2)} + \frac{4\rho^2}{4-\rho^2}. \end{aligned} \tag{19}$$

Here we have used

$$\mathrm{Cov}(\boldsymbol{\epsilon}, \mathbf{y}) = (I - \rho W)^{-1} = (A_{\rho/2} + o(\rho))^{-1} = A^{-1}_{\rho/2} + o(\rho), \tag{20}$$

where A_ϕ^{-1} is equal to

$$\begin{pmatrix} 1 & -\phi & 0 & \dots & 0 \\ -\phi & 1+\phi^2 & -\phi & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -\phi & 1+\phi^2 & -\phi \\ 0 & \dots & 0 & -\phi & 1 \end{pmatrix}^{-1} = \frac{1}{1-\phi^2} \begin{pmatrix} 1 & \phi & \phi^2 & \dots & \phi^{n-1} \\ \phi & 1 & \phi & & \vdots \\ \phi^2 & \ddots & \ddots & \ddots & \phi^2 \\ \vdots & & \phi & 1 & \phi \\ \phi^{n-1} & \dots & \phi^2 & \phi & 1 \end{pmatrix} \tag{21}$$

and similarly for

$$\mathrm{Var}(\mathbf{y}) = [(I - \rho W^T)(I - \rho W)]^{-1} = (I - 2\rho W + o(\rho))^{-1} = A_\rho^{-1} + o(\rho). \tag{22}$$

Thus, (17) requires $P_X^{-1}(\cdot)$ corresponding to a χ^2_1 random variable X. Figure 2 plots the value of π as a function of ρ for which the median under $\hat{\rho}_{LMS} = -1, 0,$ or $+1$ equals that under $\hat{\rho}_{LMS} = \rho$. The minimum of these curves is depicted in bold. The maximum of the bold curve, 14.4% at $\rho = \pm 0.714$, is the breakdown-point according to the definition above. If the fraction of contamination exceeds this

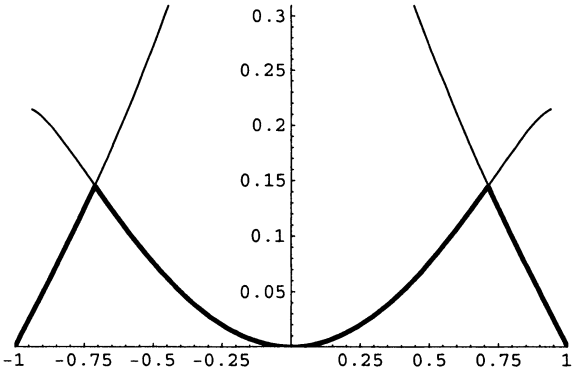

Fig. 2. Breakdown-curve (in bold) of the LMS estimator in the unidimensional Gaussian SAR of order 1, as a function of ρ.

breakdown-point, the set of badness values under the postulated extreme additive outlier process equals $\{|-1-\rho|, |\rho|, |1-\rho|\}$ for $\mathcal{Y}$ (or $\rho' \in (-1,1)$), which for given $\mathbf{y}$ (or ρ) is a finite set. For comparison, Figure 3 plots the breakdown-curve (in bold) of the LMS estimator in the Gaussian AR(1) as a function of ρ (see Genton and Lucas, 2000, for computational details). In this case, the breakdown-point according to our definition is 22.1% at $\rho = \pm 0.734$, slightly larger than for the unidimensional SAR model. This is a consequence of the AR(1) model depending only on 1 neighbor, whereas the unidimensional SAR model depends on two nearest neighbors. The breakdown-point, therefore, lies below the familiar 50% for LMS in the regression setting. It has been argued that a breakdown-point of $50\%/(1+p)$, where $p = 1$ is the order of the autoregressive process, might be more appropriate in the time series or spatial statistics setting. Figures 2 and 3, however, illustrate that the actual breakdown-point is even lower.

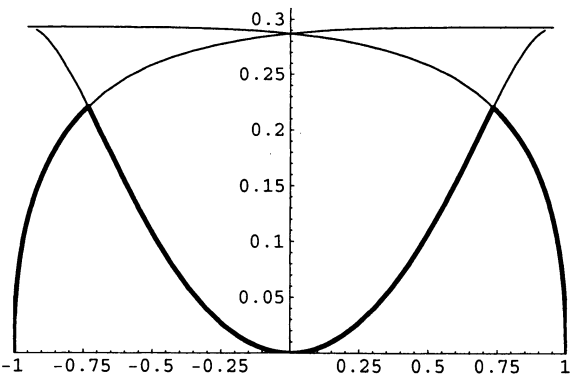

Fig. 3. Breakdown-curve (in bold) of the LMS estimator in the Gaussian AR(1) as a function of ρ.

To conclude this section, it is useful to note what happens to the ML estimator for the SAR, QAR, and AR(1) model. Under the same contaminating process, badness measured as bias, and set of alternative uncontaminated processes ($\rho' \in (-1, 1)$), we have for the AR(1)

$$\hat{\rho} = \frac{\mathrm{E}(Y_i Y_{i-1}) + \mathrm{E}(Z_i^{\zeta} Z_{i-1}^{\zeta})}{\mathrm{E}(Y_{i-1}^2) + \mathrm{E}[(Z_{i-1}^{\zeta})^2]} = O(\zeta^{-2}) \overset{\zeta \to \infty}{\longrightarrow} 0$$

for arbitrary small positive π. The badness set R^* for arbitrary small but positive π and $\zeta \to \infty$ is $\{|0 - \rho|\}$, which is a finite set for given $\mathbf{y}$ (or ρ). Therefore, the breakdown-point of the ML estimator is zero in the asymptotic AR time series setting as well. A similar line of argument applied to the SAR and QAR models shows that the ML estimator has also breakdown-point of zero in this spatial statistics context.

5 Discussion

Although the seminal definitions of tolerance and breakdown-point introduced by Hodges (1967) and Hampel (1971) are intuitively appealing, they are not suitable for situations with correlated observations. Genton and Lucas (2000) introduced a concept of breakdown-point that covers both independent and correlated observations, as well as linear and nonlinear regression settings. Their definition is more complex than the traditional ones, but this has to be expected given the huge step existing between independent and correlated observations. In this paper we have presented several examples from spatial statistics, such as SAR and QAR models, for which the new definition of breakdown-point is appropriate, whereas previous definitions typically fail. In particular, it has been shown that the LMS estimator in a spatial statistics context can have a breakdown-point much smaller than the familiar 50%. Table 1 provides a summary of available breakdown-point definitions.

Genton and Lucas (2000) provide further examples of computations of their breakdown-point concept, for example the deepest regression estimator of the AR(1), as well as relations to available notions of breakdown. Of particular interest is the Michaelis-Menten model in the context of nonlinear regression, where the least-squares estimator can have broken according to our definition, whereas it has still not broken in the sense of Stromberg and Ruppert (1992). The reason is that we leave implicit the absolute magnitude of the badness and the relative magnitude with respect to the supremum badness. In other words, the estimator may already have broken in the sense of our definition while the badness still increases upon adding additional outliers.

References

D.L. Donoho and P.J. Huber. The notion of breakdown point. In P.J. Bickel, K.A. Doksum, and J.L. Hodges, editors, *A Festschrift for Erich L. Lehmann*, pages 157–184. Wadsworth,

Table 1. Summary of available breakdown-point definitions

Author(s)	Year	Setting
Hodges	1967	Location
Hampel	1971	Parametric in general
Donoho-Huber	1983	Parametric in general
He-Simpson-Portnoy	1990	Test
Stromberg-Ruppert	1992	Nonlinear regression
Sakata-White	1995	Parametric in general; Nonlinear regression
Genton-Lucas	2000	Parametric in general; Nonlinear regression (independent and correlated observations)

Author(s)	Definition
Hodges	The estimate is arbitrarily large (positively or negatively)
Hampel	The stochastic limit of the estimator is pushed to the edge of the parameter space
Donoho-Huber	The estimate under contamination is arbitrarily far from the one under no contamination
He-Simpson-Portnoy	The test statistic under contamination tends to a null value
Stromberg-Ruppert	The fit of the model goes to its infimum or supremum under contamination
Sakata-White	A badness measure goes to its infimum or supremum under contamination
Genton-Lucas	The badness set is a finite set for (extreme) outliers

Belmont, CA, 1983.

M.G. Genton. Spatial breakdown point of variogram estimators. *Mathematical Geology*, 30: 853–871, 1998.

M.G. Genton. The change-of-variance function: a tool to explore the effects of dependencies in spatial statistics. *J. Statist. Planning and Inf.*, 98:191–209, 2001.

M.G. Genton and A. Lucas. Comprehensive definitions of breakdown-points for independent and dependent observations. Tinbergen Discussion Paper TI 2000-040/2. Submitted, 2000.

F.R. Hampel. A general qualitative definition of robustness. *Ann. Math. Statist.*, 42:1887–1896, 1971.

F.R. Hampel, E.M. Ronchetti, P.J. Rousseeuw, and W.A. Stahel. *Robust statistics: The approach based on influence functions*. Wiley, New York, 1986.

X. He. A local breakdown property of robust tests in linear regression. *J. Multivariate Statist.*, 38:294–305, 1991.

X. He, D.G. Simpson, and S.L. Portnoy. Breakdown robustness of tests. *J. Am. Statist. Assoc.*, 85:446–452, 1990.

J.L. Hodges. Efficiency in normal samples and tolerance of extreme values for some estimates of location. In *Proceedings of the Fifth Berkeley Symp. Math. Statist. and Probab.*, volume 1, pages 163–168, 1967.

O. Hössjer. The change-of-variance function for dependent data. *Probability Theory and Related Fields*, 90:447–467, 1991.

H.R. Künsch. Infinitesimal robustness for autoregressive processes. *Ann. Statist.*, 12(3): 843–863, 1984.

Y. Ma and M.G. Genton. Highly robust estimation of the autocovariance function. *J. of Time Series Analysis*, 21:663–684, 2000.

R.D. Martin. Robust estimation of autoregressive models. In D.R. Brillinger and G.C. Tiao, editors, *Directions in Time Series*, pages 228–254. Institute of Mathematical Statistics, Hayward, CA, 1980.

R.D. Martin and J. De Jong. Asymptotic properties of robust generalized M-estimates for the first-order autoregressive parameter. Technical report, Bell Laboratories Technical Memo, Murray Hill, New Jersey, 1977.

R.D. Martin and V.J. Yohai. Influence funtionals for time series. *Ann. Statist.*, 14:781–855, 1986.

K. Ord. Estimation method for models of spatial interaction. *J. Am. Statist. Assoc.*, 70: 120–126, 1975.

P.J. Rousseeuw. A new infinitesimal approach to robust estimation. *Zeit. Wahr. verw. Geb.*, 56:127–132, 1981.

P.J. Rousseeuw. Least median of squares regression. *J. Am. Statist. Assoc.*, 79:871–880, 1984.

S. Sakata and H. White. An alternative definition of finite sample breakdown point with applications to regression model estimators. *J. Am. Statist. Assoc.*, 90:1099–1106, 1995.

S. Sakata and H. White. High breakdown point conditional dispersion estimation with application to S&P 500 daily returns volatility. *Econometrica*, 66:529–567, 1998.

A.J. Stromberg and D. Ruppert. Breakdown in nonlinear regression. *J. Am. Statist. Assoc.*, 87:991–997, 1992.

P. Whittle. On stationary processes in the plane. *Biometrika*, 41:434–449, 1954.