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ON A TIME DEFORMATION REDUCING NONSTATIONARY STOCHASTIC PROCESSES TO LOCAL STATIONARITY

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Abstract

A stochastic process is locally stationary if its covariance function can be expressed as the product of a positive function multiplied by a stationary covariance. In this paper, we characterize nonstationary stochastic processes that can be reduced to local stationarity via a bijective deformation of the time index, and we give the form of this deformation under smoothness assumptions. This is an extension of the notion of stationary reducibility. We present several examples of nonstationary covariances that can be reduced to local stationarity. We also investigate the particular situation of exponentially convex reducibility, which can always be achieved for a certain class of separable nonstationary covariances.

Keywords: Exponentially convex; local stationarity; nonstationarity; positive-definite function; reducibility; stationarity; separable covariance function; stochastic processes; time series

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1. Introduction

In the last few decades, applications have shown that stationary phenomena are more often the exception rather than the rule. A classical approach to the modelling of a nonstationary stochastic process $Z = \{Z(x),\ x \in T \subseteq \mathbb{R}\}$ consists in defining

$$Z(x) = \mu(x) + \sigma(x)\varepsilon(x),$$

where $\mu(x) = \mathrm{E}(Z(x))$, $\sigma^2(x) = \mathrm{var}(Z(x))$, and $\varepsilon(x)$ is a weakly stationary process with expectation zero and unit variance. Usually, the nonstationarity of Z is a consequence of the nonstationarity of both the expectation $\mu(x)$ and the variance $\sigma^2(x)$. Stock (1988) and Sampson and Guttorp (1992) introduced further nonstationarity by modelling $\varepsilon(x)$ as $\varepsilon(x) = \delta(f(x))$, where δ is a weakly stationary process and f is a bijective deformation. This is also equivalent to modelling the nonstationary covariance function $\rho(x, y)$ of the process ε as

$$\rho(x, y) = R(f(x) - f(y)), \tag{1}$$

where R is a stationary covariance function. Sampson and Guttorp (1992), as well as Meiring (1995) and Perrin (1997), further developed this model in the case of spatial random fields.

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The model (1) allows us to take into account second-order nonstationarities and thus provides a rich class of nonstationary processes.

A covariance function ρ satisfying (1) is said to be stationary reducible (SR). Perrin and Senoussi (1999) characterized stationary reducibility under smoothness assumptions on ρ and f. In particular, they showed that a covariance function ρ satisfies (1) if and only if, for some arbitrarily chosen point $x_0 \in T$,

$$\rho^{(1,0)}(x, y)\frac{\rho^{(1,0)}(y, x_0)}{\rho^{(0,1)}(y, x_0)} + \rho^{(0,1)}(x, y)\frac{\rho^{(1,0)}(x, x_0)}{\rho^{(0,1)}(x, x_0)} = 0, \tag{2}$$

where $x \neq y$ almost everywhere and $\rho^{(m,m')}(x, y)$ denotes the (m, m')-partial derivative of ρ with respect to x and y. Moreover, the pair (f, R) in (1) is uniquely determined by

$$f(x) = -\int_{x_0}^{x} \frac{\rho^{(1,0)}(y, x_0)}{\rho^{(0,1)}(y, x_0)}\,\mathrm{d}y \quad \text{and} \quad R(u) = \rho(x_0, f^{-1}(u)).$$

For example, the correlation function of any H-self-similar process with index $H \in]0, 1]$, such as nondegenerate fractional Brownian motion, satisfies (2), and is therefore SR with $f(x) = \ln x$. Nevertheless, not all nonstationary processes can be reduced to stationarity through a deformation f of the time axis. For example, the nonstationary covariance

$$r(x, y) = \exp(-x^3 - y^3) \tag{3}$$

is not SR since it does not satisfy (2). Moreover, reducibility to stationarity means implicitly that the variance of the original process is constant, otherwise its standardized version must be computed and we need to check whether the correlation of the process satisfies (2). Our motivation in this paper is to extend the model (1) of Sampson and Guttorp (1992) to processes with a variance which is not necessarily constant and thus to propose a wider family of nonstationary models which includes Sampson and Guttorp's model as a subfamily. For this purpose, we consider a class of nonstationary processes, first introduced by Silverman (1957), called locally stationary (LS) processes. They were introduced to describe physical systems for which statistical characteristics change slowly in time. Specifically, a stochastic process Y is said to be LS, in the weak sense, if its covariance function c can be written in the form

$$c(x, y) = R_1\left(\frac{x+y}{2}\right)R_2(x - y), \tag{4}$$

where R_2 is a stationary covariance function. The property $c(x, x) \geq 0$ forces the function R_1 to be nonnegative. A general example of such a process $Y = \{Y(x),\ x \in T \subseteq \mathbb{R}\}$ can be given by taking Y as the product of two independent and centred processes η and ν: $Y(x) = \eta(x)\nu(x)$, where η has covariance $r_1(x, y) = R_1((x + y)/2)$ and ν is stationary with covariance $r_2(x, y) = R_2(x - y)$. Indeed, in this example,

$$\begin{aligned} c(x, y) &= \operatorname{cov}(Y(x), Y(y)) \\ &= \operatorname{cov}(\eta(x)\nu(x), \eta(y)\nu(y)) \\ &= \mathrm{E}(\eta(x)\eta(y))\,\mathrm{E}(\nu(x)\nu(y)) \\ &= R_1\left(\frac{x+y}{2}\right)R_2(x - y). \end{aligned}$$

The variable $(x + y)/2$ has been chosen because of its suggestive meaning of the average or centroid of the time points x and y. Indeed, when R_1 is smooth enough, $c(x, y) \simeq R_2(x - y)$ when $y \in [x - \varepsilon_x/2, x + \varepsilon_x/2]$ for ε_x small enough. This justifies the LS structure of Y. Note that, if R_1 is a positive constant, then (4) reduces to a stationary covariance. Thus, the class of LS covariance functions has the desirable property of including stationarity as a special case. If the covariance function c can be written as in (4) but R_2 is not a stationary covariance function, then we call it pseudo-locally stationary (PLS). Note that the product of a positive function R_1 by a stationary covariance function R_2 as in (4) does not necessarily yield a positive-definite covariance $c(x, y)$. However, R_1 and R_2 can be simultaneously modified in order to make $c(x, y)$ positive definite; see the matching theorem in Silverman (1959).

We generalize Sampson and Guttorp's model by defining locally stationary reducible (LSR) processes. A process $Z = \{Z(x),\ x \in T \subseteq \mathbb{R}\}$ is said to be LSR if $Z(x) = Y(g(x))$, where Y is an LS process and g a bijective deformation of the time index T. This is also equivalent to modelling the nonstationary covariance function r of the process Z as

$$r(x, y) = R_1\left(\frac{g(x) + g(y)}{2}\right) R_2(g(x) - g(y)), \tag{5}$$

where R_1 is a nonnegative function and R_2 is a stationary covariance function. Note that, if R_1 is a positive constant, then (5) reduces to the model (1) of Sampson and Guttorp (1992). Here again, if R_2 is not a stationary covariance, we say that Z is only pseudo-locally stationary reducible (PLSR).

Notice that local stationarity is a general concept for which different definitions exist. Berman (1974) defined it in terms of Hölder conditions. Priestley (1965), and further Dahlhaus (1997), considered a process with continuously time-changing spectral representation. Recently, a different and less restrictive notion of local stationarity has been introduced by means of orthogonal wavelets (see e.g. Mallat *et al.* (1998)). However, we restrict ourselves to the definition of Silverman (1957), which is more tractable in the context of continuous-time deformation. With additional smoothness assumptions on R_1 and R_2, Silverman's definition of local stationarity can be viewed as a particular case of the later versions.

The remainder of the paper is set up as follows. In Section 2, we discuss the properties of LS processes and give a characterization of such processes. The characterization of locally stationary reducibility is given in Section 3 and two examples of LSR processes are presented in Section 4. In Section 5, we treat the particular case when R_2 is a constant, and we conclude the paper with a discussion in Section 6.

2. Locally stationary processes

2.1. Properties

Because the product of R_1 and R_2 in (4) is defined only up to a multiplicative positive constant, we further impose without loss of generality the condition that $R_2(0) = 1$. Moreover, we can always impose the condition that $c(x_0, x_0) = 1$ for some $x_0 \in T$. Indeed, it is always possible to deal with the covariance $c(x, y)/c(x_0, x_0)$ instead. From now on we always assume that the following condition is satisfied.

Condition 1. *For some arbitrarily chosen point* $x_0 \in T$, $c(x_0, x_0) = 1$ *and* $R_2(0) = 1$.

From Condition 1, we get directly that $R_1(x_0) = 1$ and that the variance of $Y(x)$ is

$$\operatorname{var}(Y(x)) = c(x, x) = R_1(x) R_2(0) = R_1(x), \tag{6}$$

thus justifying the name of power schedule for R_1, which describes the global structure of Y, while R_2 describes the local stationarity of Y, as already mentioned in Section 1. Using this property, we can also define R_2 directly from c by considering

$$c(x_0 + \tfrac{1}{2}x, x_0 - \tfrac{1}{2}x) = R_1(x_0)R_2(x) = R_2(x). \tag{7}$$

Equations (6) and (7) imply that the covariance function $c(x, y)$ defined by (4) is completely determined by its values on the diagonal $x = y$ and the antidiagonal $y = -x + 2x_0$ in the plane, for

$$c(x, y) = c\left(\frac{x+y}{2}, \frac{x+y}{2}\right)c\left(x_0 + \frac{x-y}{2}, x_0 - \frac{x-y}{2}\right).$$

Note that $R_1((x + y)/2)$ is invariant with respect to shifts parallel to the antidiagonal, whereas $R_2(x - y)$ is invariant with respect to shifts parallel to the diagonal.

2.2. Characterization

We make the following assumptions.

Assumption 1. *There is at most a countable set of points* (x, y) *in* T^2 *such that* $c(x, y) = 0$.

Assumption 2. *The function* c *is continuous in* T^2 *and has second derivatives which are uniformly bounded for* $x \neq y$.

First let us characterize the separable covariance functions, i.e. the covariance functions of the form

$$\gamma(u, v) = c(u + \tfrac{1}{2}v, u - \tfrac{1}{2}v) = R_1(u)R_2(v) \tag{8}$$

for all (u, v).

Lemma 1. *Assume that* γ *is twice differentiable for* $u \neq v$. *Then* γ *is separable if and only if the following holds for* $u \neq v$ *up to a countable set of points such that* $\gamma(u, v) \neq 0$:

$$\gamma^{(1,1)}(u, v)\gamma(u, v) = \gamma^{(1,0)}(u, v)\gamma^{(0,1)}(u, v). \tag{9}$$

Proof. The necessity is obvious. Conversely, assume that (9) holds. Then the derivative of $\gamma^{(1,0)}(u, v)/\gamma(u, v)$ with respect to v is equal to 0. Therefore, we may define

$$\frac{\gamma^{(1,0)}(u, v)}{\gamma(u, v)} = k_1(u).$$

By integration with respect to u we obtain that $\ln|\gamma(u, v)| = K_1(u) + K_2(v)$ everywhere for $u \neq v$ such that $\gamma(u, v) \neq 0$, with K_1 being a primitive function of k_1. To conclude, we set $\exp(K_1(u)) = R_1(u)$ and $\exp(K_2(v)) = |R_2(v)|$.

Considering $u \neq v$ allows us to include wide classes of covariance functions which are not necessarily differentiable on the diagonal and which characterize the so-called nugget-effect phenomenon in geostatistics. Here is a necessary and sufficient condition for a covariance to have the representation (4) with R_2 a possibly nonstationary covariance function.

Theorem 1. *Let Assumptions 1 and 2 hold. A covariance* c *is PLS, i.e. of the form (4), if and only if the following holds for* $x \neq y$ *up to a countable set of points in* T^2 *such that* $c(x, y) \neq 0$:

$$(c^{(2,0)}(x, y) - c^{(0,2)}(x, y))c(x, y) = (c^{(1,0)}(x, y) - c^{(0,1)}(x, y))(c^{(1,0)}(x, y) + c^{(0,1)}(x, y)).$$

Proof. We set $\gamma(u, v) = c(u+v/2, u-v/2)$. Thus c satisfies (4) if and only if γ satisfies (8). It follows from Lemma 1 that c satisfies (4) if and only if (9) holds with

$$\gamma^{(1,0)}(u, v) = c^{(1,0)}\left(u + \frac{v}{2}, u - \frac{v}{2}\right) + c^{(0,1)}\left(u + \frac{v}{2}, u - \frac{v}{2}\right),$$

$$\gamma^{(0,1)}(u, v) = \frac{1}{2}\left(c^{(1,0)}\left(u + \frac{v}{2}, u - \frac{v}{2}\right) - c^{(0,1)}\left(u + \frac{v}{2}, u - \frac{v}{2}\right)\right),$$

$$\gamma^{(1,1)}(u, v) = \frac{1}{2}\left(c^{(2,0)}\left(u + \frac{v}{2}, u - \frac{v}{2}\right) - c^{(0,2)}\left(u + \frac{v}{2}, u - \frac{v}{2}\right)\right).$$

To conclude, we set $x = u + v/2$ and $y = u - v/2$.

Corollary 1. *Let Assumptions 1 and 2 hold. A covariance c is PLS if and only if $l(x, y) = \ln|c(x, y)|$ satisfies the following wave equation up to a countable set of points in T^2 such that $c(x, y) \neq 0$:*

$$l^{(2,0)}(x, y) = l^{(0,2)}(x, y).$$

Proof. This is obvious by the definition of $l(x, y)$ since

$$l(x, y) = \ln\left(R_1\left(\frac{x+y}{2}\right)\right) + \ln|R_2(x - y)|.$$

Note that a covariance function is locally stationary if and only if it is PLS and R_2 is a stationary covariance function.

2.3. Examples

A large class of LS covariances can be constructed by multiplying two covariances $R_1((x + y)/2)$ and $R_2(x - y)$. Covariances of the form $R_1((x + y)/2)$ were studied by Loève (1946) who called them exponentially convex covariances. However, the product $R_1((x + y)/2)R_2(x - y)$ can be a covariance without $R_1((x + y)/2)$ being a covariance function. The following two examples given by Silverman (1957) are LS covariances which are not the product of two covariances. The first example is

$$c(x, y) = \exp[-a(x^2 + y^2)] = \exp\left[-2a\left(\frac{x+y}{2}\right)^2\right]\exp\left[-a\frac{(x-y)^2}{2}\right], \qquad a > 0,$$

where the first factor in the right-hand side is a positive function without being a covariance, and the second factor is a covariance. Secondly, with the positive-definite delta covariance $\delta(x - y)$, which is equal to 1 if $x = y$ and 0 otherwise, the product

$$c(x, y) = R_1\left(\frac{x+y}{2}\right)\delta(x - y)$$

is an LS covariance provided that R_1 is any nonnegative function, not necessarily a covariance. A process with such a covariance is called a locally stationary white noise.

If R_2 reduces to the constant 1, (4) reduces to the exponentially convex covariance

$$c(x, y) = R_1\left(\frac{x+y}{2}\right). \tag{10}$$

From (10) we get that $R_1((x + y)/2) = c((x + y)/2, (x + y)/2) \geq 0$. Actually, as noted by Loève, any two-sided Laplace transform of a nonnegative function is an exponentially convex covariance. For instance, the two-sided Laplace transform of the standard normal density yields the following exponentially convex covariance:

$$R_1(x, y) = \exp\left(\frac{(x+y)^2}{2}\right).$$

Conversely, Loève noted that any continuous exponentially convex covariance can be represented as the two-sided Laplace transform of a nonnegative function in the absolutely continuous case. Ehm *et al.* (2003) established a bijection between exponentially convex functions and entire positive-definite functions. As an application, they derived parametric models of exponentially convex and locally stationary covariances.

Under smoothness assumptions, another characterization can be given by the following lemma for which the proof can be omitted due to its simplicity.

Lemma 2. *Let $c(x, y)$ be a covariance function which is differentiable. Then $c(x, y)$ is an exponentially convex covariance, i.e. satisfies (10), if and only if the following holds for all (x, y):*

$$c^{(1,0)}(x, y) = c^{(0,1)}(x, y).$$

Finally, we give an example of a covariance which is PLS but not LS. Consider the covariance

$$c(x, y) = \exp(x^2 + y^2 + xy - 3), \tag{11}$$

which is PLS by Corollary 1. Indeed, this covariance is of the form (4) with

$$R_1(u) = \exp(3u^2 - 3),$$

$$R_2(v) = \exp\left(\frac{v^2}{4}\right).$$

Since R_2 is not positive definite, it is not a stationary covariance function, so that (11) is not an LS covariance, but only PLSR.

3. Locally stationary reducibility

In this section we characterize covariance functions that can be reduced to local stationarity via a bijective deformation of the time index, i.e. covariance functions of the form (5).

3.1. Main result

We consider the following assumption on the deformation g.

Assumption 3. *The deformation g is continuous and twice differentiable in T as is its inverse g^{-1} in $g(T)$.*

We assume that Assumptions 1 and 2 hold for r. Since g is a bijection, it is either increasing or decreasing. Remarking that g increasing is equivalent to $-g$ decreasing, we can restrict to g increasing without loss of generality. Moreover, note that if (g, R_1, R_2) is a solution of (5), then, for any $\beta \in \mathbb{R}$, $\alpha > 0$, and $\lambda > 0$, $(\tilde{g}, \tilde{R}_1, \tilde{R}_2)$ is a solution as well, where $\tilde{g}(x) = \alpha g(x) + \beta$, $\tilde{R}_1(u) = \lambda R_1((u - \beta)/\alpha)$, and $\tilde{R}_2(v) = R_2(v/\alpha)/\lambda$. Thus, without loss of generality, in addition to Condition 1, we may impose in the sequel the following restriction.

Condition 2. *We have $g(x_0) = 0$ and $g^{(1)}(x_0) = 1$.*

Here, $g^{(1)}$ is the first derivative of g. Consequently, when the nonstationary covariance r satisfying (5) and the deformation g are given, the power schedule R_1 and the stationary covariance function R_2 are uniquely determined by

$$R_1(u) = r(g^{-1}(u), g^{-1}(u)), \tag{12}$$

$$R_2(v) = r\left(g^{-1}\left(\frac{v}{2} + x_0\right), g^{-1}\left(-\frac{v}{2} + x_0\right)\right). \tag{13}$$

It follows from Assumptions 2 and 3 that R_1 and R_2 are continuous and differentiable except maybe at the origin. Before giving the necessary and sufficient condition for locally stationary reducibility via bijective deformation we give the following lemma.

Lemma 3. *Let Assumptions 1–3 hold. A covariance function r is PLSR, i.e. satisfies (5), if and only if the following holds for $x \neq y$ up to a countable set of points in T^2 such that $r(x, y) \neq 0$:*

$$\begin{aligned} &\Phi(x)(\ln|r(x, y)|)^{(2,0)} + \tfrac{1}{2}\Phi^{(1)}(x)(\ln|r(x, y)|)^{(1,0)} \\ &\quad = \Phi(y)(\ln|r(x, y)|)^{(0,2)} + \tfrac{1}{2}\Phi^{(1)}(y)(\ln|r(x, y)|)^{(0,1)}, \end{aligned} \tag{14}$$

where

$$\Phi(x) = \left(\frac{1}{g^{(1)}(x)}\right)^2.$$

Proof. We set $c(u, v) = r(g^{-1}(u), g^{-1}(v))$. Thus, r satisfies (5) if and only if c satisfies (4). It follows from Theorem 1 that r satisfies (5) if and only if, for $u \neq v$ up to a countable set of points in $(g(T))^2$ such that $c(u, v) \neq 0$,

$$(c^{(2,0)}(u, v) - c^{(0,2)}(u, v))c(u, v) = (c^{(1,0)}(u, v) - c^{(0,1)}(u, v))(c^{(1,0)}(u, v) + c^{(0,1)}(u, v)),$$

with

$$c^{(1,0)}(u, v) = \frac{r^{(1,0)}(g^{-1}(u), g^{-1}(v))}{g^{(1)}(g^{-1}(u))},$$

$$c^{(0,1)}(u, v) = \frac{r^{(0,1)}(g^{-1}(u), g^{-1}(v))}{g^{(1)}(g^{-1}(v))},$$

$$c^{(2,0)}(u, v) = \frac{r^{(2,0)}(g^{-1}(u), g^{-1}(v))}{(g^{(1)}(g^{-1}(u)))^2} - \frac{r^{(1,0)}(g^{-1}(u), g^{-1}(v))g^{(2)}(g^{-1}(u))}{(g^{(1)}(g^{-1}(u)))^3},$$

$$c^{(0,2)}(u, v) = \frac{r^{(0,2)}(g^{-1}(u), g^{-1}(v))}{(g^{(1)}(g^{-1}(v)))^2} - \frac{r^{(0,1)}(g^{-1}(u), g^{-1}(v))g^{(2)}(g^{-1}(v))}{(g^{(1)}(g^{-1}(v)))^3},$$

where $g^{(2)}$ is the second derivative of g. We set $x = g^{-1}(u)$ and $y = g^{-1}(v)$ to obtain that r satisfies (5) if and only if

$$\begin{aligned} &\frac{1}{(g^{(1)}(x))^2}(r^{(2,0)}(x, y)r(x, y) - (r^{(1,0)}(x, y))^2) - \frac{g^{(2)}(x)}{(g^{(1)}(x))^3}r^{(1,0)}(x, y)r(x, y) \\ &\quad = \frac{1}{(g^{(1)}(y))^2}(r^{(0,2)}(x, y)r(x, y) - (r^{(0,1)}(x, y))^2) - \frac{g^{(2)}(y)}{(g^{(1)}(y))^3}r^{(0,1)}(x, y)r(x, y). \end{aligned}$$

We multiply this equation by $1/(r(x, y))^2$ and set $\Phi(x) = (1/g^{(1)}(x))^2$ to conclude.

We set

$$r(x_0^+, x_0) = \lim_{x \searrow x_0} r(x, x_0) \quad \text{and} \quad r(x_0^-, x_0) = \lim_{x \nearrow x_0} r(x, x_0).$$

The following theorem is our main result.

Theorem 2. *Let Assumptions 1–3 hold. A covariance function r is PLSR, i.e. satisfies (5), if and only if the following holds for $x \neq y$ up to a countable set of points in T^2 such that $r(x, y) \neq 0$:*

$$\begin{aligned}\Phi(x)(\ln|r(x,y)|)^{(2,0)} &+ \tfrac{1}{2}\Phi^{(1)}(x)(\ln|r(x,y)|)^{(1,0)} \\ &= \Phi(y)(\ln|r(x,y)|)^{(0,2)} + \tfrac{1}{2}\Phi^{(1)}(y)(\ln|r(x,y)|)^{(0,1)},\end{aligned}$$

with

$$\Phi(x) = \begin{cases} \dfrac{((\ln|r(x_0^+, x_0)|)^{(1,0)})^2 + k(x,\theta)}{((\ln|r(x,x_0)|)^{(1,0)})^2} & \text{if } x \geq x_0, \\ \dfrac{((\ln|r(x_0^-, x_0)|)^{(1,0)})^2 + k(x,\theta)}{((\ln|r(x,x_0)|)^{(1,0)})^2} & \text{if } x < x_0, \end{cases} \tag{15}$$

where

$$k(x,\theta) = 2\int_{x_0}^{x} (\ln|r(s,x_0)|)^{(1,0)}((\ln|r(s,x_0)|)^{(0,2)} - \theta(\ln|r(s,x_0)|)^{(0,1)})\,\mathrm{d}s \tag{16}$$

and θ is the solution of the equation

$$\left[\left(\frac{1}{\Phi(x)}\right)^{(1)}\right]_{x=x_0} = 2\theta.$$

The triplet (g, R_1, R_2) in (5) is uniquely determined with g given by

$$g(x) = \int_{x_0}^{x} \frac{1}{\sqrt{\Phi(s)}}\,\mathrm{d}s, \tag{17}$$

and R_1 and R_2 given by (12) and (13). Moreover, if R_2 is a stationary covariance function, then r is LSR.

Proof. In (14), we set $y = x_0$ so that, for $x \neq x_0$, we get the following inhomogeneous second-order differential equation with respect to Φ:

$$\begin{aligned}\Phi(x)(\ln|r(x,x_0)|)^{(2,0)} &+ \tfrac{1}{2}\Phi^{(1)}(x)(\ln|r(x,x_0)|)^{(1,0)} \\ &= \Phi(x_0)(\ln|r(x,x_0)|)^{(0,2)} + \tfrac{1}{2}\Phi^{(1)}(x_0)(\ln|r(x,x_0)|)^{(0,1)},\end{aligned}$$

with $\Phi(x) = (1/g^{(1)}(x))^2$. Under Condition 2, we then have $\Phi(x_0) = 1$ and $\Phi^{(1)}(x_0) = -2\theta$, where θ denotes the second derivative of g at x_0. Solving this inhomogeneous differential equation is standard and can be carried out with the variation of the constants method (see for instance Collatz (1986, pp. 13–17)). Using this method leads directly to the solution given by (15), and the rest of the proof follows directly.

3.2. An illustrative example

We now illustrate with an example the constructive feature of Theorem 2. In a first step, we construct a non-LS covariance function r via a time deformation g satisfying Assumption 3 and an LS covariance function c such that r satisfies Assumptions 1 and 2. In a second step, we show the reverse process of recovering the transformation g from r, as well as the LS covariance function c.

(i) We consider an LS random process $Y = \{Y(u), u \in \mathbb{R}^+\}$ with covariance function

$$c(u, v) = \exp\left(\frac{(u+v)^2}{2}\right)\exp(-|u-v|),$$

and the transformation $g : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$g(x) = \ln(1+x),$$

for which Assumption 3 holds. Starting from c and g, we get

$$r(x, y) = \exp\left(\frac{(\ln(1+x)+\ln(1+y))^2}{2}\right)\exp(-|\ln(1+x)-\ln(1+y)|).$$

It is easy to check that Assumptions 1 and 2 are satisfied. In addition, Conditions 1 and 2 are fulfilled with $x_0 = 0$.

(ii) Conversely, we want to recover the transformation g and the LS covariance c satisfying $r(x, y) = c(g(x), g(y))$ from the non-LS covariance function r. We suppose that $x \geq y$.

To start, we compute the first derivatives of $\ln(|r(x, y)|) = \ln(r(x, y))$ as well as its second derivative with respect to y:

$$\begin{aligned}
(\ln(|r(x,y)|))^{(1,0)} &= \frac{\ln(1+x)+\ln(1+y)-1}{1+x},\\
(\ln(|r(x,y)|))^{(0,1)} &= \frac{\ln(1+x)+\ln(1+y)+1}{1+y},\\
(\ln(|r(x,y)|))^{(0,2)} &= -\frac{\ln(1+x)+\ln(1+y)}{(1+y)^2}.
\end{aligned}$$

From this we can deduce the individual terms in (15) and (16):

$$\begin{aligned}
(\ln(|r(0^+,0)|))^{(1,0)} &= -1,\\
(\ln(|r(x,0)|))^{(1,0)} &= \frac{\ln(1+x)-1}{1+x},\\
(\ln(|r(x,0)|))^{(0,1)} &= \ln(1+x)+1,\\
(\ln(|r(x,0)|))^{(0,2)} &= -\ln(1+x).
\end{aligned}$$

Then, using the fact that, for any $p \in \mathbb{N}^*$, $((\ln(1+x))^{p+1})/(p+1)$ is a primitive function of $(\ln(1+x))^p/(1+x)$, we compute the function Φ in (15):

$$\Phi(x) = \frac{(1+x)^2(1+2\theta\ln(1+x)+(\ln(1+x))^2-\frac{2}{3}(1+\theta)(\ln(1+x))^3)}{(\ln(1+x)-1)^2}.$$

Now we compute the derivative of $1/\Phi$ at 0:

$$\left[\left(\frac{1}{\Phi(x)}\right)^{(1)}\right]_{x=0} = -4 - 2\theta.$$

Solving the equation $-4 - 2\theta = 2\theta$ leads to $\theta = -1$. Setting $\theta = -1$ in (17), we obtain

$$g(x) = \int_0^x \sqrt{\frac{(\ln(1+s)-1)^2}{(1+s)^2(\ln(1+s)-1)^2}}\,\mathrm{d}s = \int_0^x \frac{1}{1+s}\,\mathrm{d}s = \ln(1+x).$$

Finally, c is given by (12) and (13).

For $x < y$, the same computations hold.

4. Two examples of LSR processes

In this section we present two illustrative examples of covariances which are LSR.

4.1. A general example

Consider an LSR process $Z = \{Z(x),\ x \in T \subseteq \mathbb{R}\}$. By definition, its covariance function r is of the form (5). A general example of such a process can be obtained by taking Z as the product of two independent and centred processes ξ and ζ: $Z(x) = \xi(x)\zeta(x)$, where ξ is exponentially convex reducible with covariance $r_1(x, y) = R_1((g(x) + g(y))/2)$ and ζ is stationary reducible (see Section 5 for a definition) with covariance $r_2(x, y) = R_2(g(x) - g(y))$. Indeed,

$$\begin{aligned} r(x, y) &= \operatorname{cov}(Z(x), Z(y)) \\ &= \operatorname{cov}(\xi(x)\zeta(x), \xi(y)\zeta(y)) \\ &= \mathrm{E}(\xi(x)\xi(y))\,\mathrm{E}(\zeta(x)\zeta(y)) \\ &= R_1\left(\frac{g(x)+g(y)}{2}\right) R_2(g(x) - g(y)). \end{aligned}$$

4.2. Fractional Brownian motion

Consider a nondegenerate fractional Brownian motion, i.e. a self-similar process where $H \in\ ]0, 1]$ is the Hurst coefficient. Its covariance is given by the following expression for all $x \geq 0$ and $y \geq 0$:

$$r(x, y) = \frac{x^{2H} + y^{2H} - |x-y|^{2H}}{2}. \tag{18}$$

For all $x > 0$ and $y > 0$,

$$r(x, y) = x^H y^H \frac{x^{2H} + y^{2H} - |x-y|^{2H}}{2x^H y^H}.$$

We know from Perrin and Senoussi (1999) that $(x^{2H} + y^{2H} - |x-y|^{2H})/2x^H y^H$ is stationary reducible, i.e.

$$\frac{x^{2H} + y^{2H} - |x-y|^{2H}}{2x^H y^H} = R_2(g(x) - g(y)),$$

where $g(x) = \ln(x)$ and $R_2(v) = \cosh(Hv) - 2^{2H-1}(\sinh(|v|/2))^{2H}$. Thus

$$r(x, y) = x^H y^H R_2(g(x) - g(y))$$

with $x^H y^H = \exp(2H(\ln(x) + \ln(y))/2)$. Therefore, r is LSR because, for all $x > 0$ and $y > 0$, we can write

$$r(x, y) = R_1\left(\frac{g(x) + g(y)}{2}\right) R_2(g(x) - g(y)),$$

with $R_1(u) = \exp(2Hu)$ and R_2 and g as given previously. For $x = 0$ or $y = 0$, we set $r(x, y) = 0$ to complete the definition (18).

5. Exponentially convex reducibility

We characterize the nonstationary covariances r that can be reduced to exponentially convex ones via a bijective deformation h, i.e.

$$r(x, y) = R_1\left(\frac{h(x) + h(y)}{2}\right), \tag{19}$$

where h satisfies Condition 2. The equation (19) is equivalent to (5) when R_2 reduces to the constant 1. Following the characterization of stationary reducible covariances by Perrin and Senoussi (1999, Theorem 2.1), we directly obtain the following characterization of exponentially convex reducible (ECR) covariances.

Theorem 3. *Let r be a continuous and differentiable covariance function in T^2 such that $r^{(1,0)}(x, x_0)/r^{(0,1)}(x, x_0)$ is locally Lebesgue integrable in T, and let h be a bijective deformation of the time index T such that both h and its inverse are continuous and differentiable in T and $h(T)$ respectively. The covariance r is ECR, i.e. satisfies (19), if and only if the following holds almost everywhere for all $(x, y) \in T^2$:*

$$r^{(1,0)}(x, y)\frac{r^{(1,0)}(y, x_0)}{r^{(0,1)}(y, x_0)} - r^{(0,1)}(x, y)\frac{r^{(1,0)}(x, x_0)}{r^{(0,1)}(x, x_0)} = 0. \tag{20}$$

Moreover, the pair (h, R_1) in (19) is uniquely determined by

$$h(x) = \int_{x_0}^{x} \frac{r^{(1,0)}(y, x_0)}{r^{(0,1)}(y, x_0)}\,\mathrm{d}y \quad \text{and} \quad R_1(u) = r(x_0, h^{-1}(2u)). \tag{21}$$

Proof. This is similar to the proof given for Theorem 2.1 of Perrin and Senoussi (1999). Note that the only difference with the latter is the factor $\frac{1}{2}$ in the definition of R_1, as well as a change of sign in (20) and (21).

An example of ECR covariances is given by separable covariances of the form

$$r(x, y) = C(x)C(y) \tag{22}$$

for all $(x, y) \in T^2$, where C is a strictly positive function which is continuous and differentiable in T as is its inverse in $C(T)$. Indeed, (22) is equivalent to

$$r(x, y) = \exp\left(2\frac{\ln(C(x)) + \ln(C(y))}{2}\right) \tag{23}$$

and thus is of the form (19) with $R_1(u) = \exp(2u)$ and $h(x) = \ln(C(x))$. Note that, if $C(x) = 0$ for at most a countable set of points $\Delta \in T$, then (23) remains true for $(x, y) \in (T\setminus\Delta)\times(T\setminus\Delta)$, and we set $r(x, y) = 0$ when $x \in \Delta$ or $y \in \Delta$ to complete the definition (22). When C takes negative values we may deal with $|r(x, y)|$. Finally, using the fact that, if r is a covariance function, then $\exp(r)$ is also a covariance function (see for instance Cristianini and Shawe-Taylor (2000)), models of the form

$$r(x, y) = \exp(C(x)C(y))$$

for all $(x, y) \in T^2$, with C satisfying the same assumptions as in (22), are also of the form (19) with $R_1(u) = \exp(\exp(2u))$ and $h(x) = \ln(C(x))$. We therefore have the two following results.

Corollary 2. *Any separable covariance of the form (22) is ECR.*

Corollary 3. *The exponential of any separable covariance of the form (22) is ECR.*

We now give three explicit examples of the type (22). Our first explicit example, again with $T = \mathbb{R}^+$, is given by the covariance function

$$r(x, y) = \frac{1}{1 + x^2 + y^2 + x^2 y^2} = \frac{1}{1 + x^2} \frac{1}{1 + y^2}, \tag{24}$$

so that $C(x) = 1/(1 + x^2) > 0$. Thus,

$$r(x, y) = \exp\left(2 \frac{\ln(C(x)) + \ln(C(y))}{2}\right) = \exp\left(-2 \frac{\ln(1 + x^2) + \ln(1 + y^2)}{2}\right),$$

which means that (24) is of the form (19) with $R_1(u) = \exp(-2u)$ and $h(x) = \ln(1 + x^2)$. Our second explicit example, with $T = \mathbb{R}^+$, is given by the covariance function

$$r(x, y) = \exp(2 + x^2 + y^2) = \exp(1 + x^2)\exp(1 + y^2), \tag{25}$$

so that $C(x) = \exp(1 + x^2) > 0$. Thus,

$$r(x, y) = \exp\left(2 \frac{\ln(C(x)) + \ln(C(y))}{2}\right) = \exp\left(2 \frac{(1 + x^2) + (1 + y^2)}{2}\right),$$

which means that (25) is of the form (19) with $R_1(u) = \exp(2u)$ and $h(x) = 1 + x^2$. Finally, our last example is given by the covariance (3) with $T = \mathbb{R}$. Obviously, this covariance is exponentially stationary reducible with $R_1(u) = \exp(-2u)$ and $h(x) = x^3$.

6. Conclusion

In this paper, we have characterized nonstationary stochastic processes that can be reduced to local stationarity via a bijective deformation of the time index, and we gave the form of this deformation under smoothness assumptions. It is then natural to ask how to estimate this time deformation from data. In the case of stationary reducibility, several suggestions can be found in the literature, e.g. based on multidimensional scaling (Sampson and Guttorp (1992)), on radial basis deformation (Perrin and Monestiez (1999)), and on simulated annealing (Iovleff and Perrin (2004)). Research is currently underway on extensions of these techniques to locally stationary reducibility.

An important advantage of PLS and LS covariance functions is that they are separable. Therefore, the full covariance matrix can be written as a Hadamard product of two matrices, one in the coordinates $u = (g(x) + g(y))/2$ and the other in the coordinates $v = g(x) - g(y)$. This is especially useful in reducing computational burden when dealing with very large datasets. Indeed, the memory requirement for the computation of the full covariance matrix for a sample of size n is reduced from $n(n + 1)/2$ to $2n - 1$ for R_1 and to n for R_2. Further computational savings can be obtained if R_2 is compactly supported; see Genton (2001).

There are also two natural extensions of the LS model (4) and its reducibility model (5). The first generalization deals with nonstationary covariances that are constant not only on the main diagonal and antidiagonal, but also on certain lines, such as

$$c(x, y) = R_0(x - y) \prod_{i=1}^{p} R_i(a_i x + b_i y), \tag{26}$$

where R_0 is a stationary covariance and $a_i, b_i \in \mathbb{R}$. We say that covariances of the form (26) are multilocally stationary. A second generalization deals of course with spatial and spatio-temporal data. In this case, the coordinates x and y in the model (4) are spatial and spatio-temporal respectively. Several different models can then be considered depending on the isotropy or anisotropy of the underlying process. Characterization and estimation of spatial deformations that reduce a nonstationary process to stationarity are currently the subject of extensive investigations because of their important application to environmental sciences. The same is also true for locally stationary reducibility and exponentially convex reducibility since they provide simple and intuitive departures from stationarity.

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