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A note on an equivalence between chi-square and generalized skew-normal distributions

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Abstract

In this note, we establish an equivalence between chi-square and generalized skew-normal distributions. This result is based on a distributional invariance property of even functions in generalized skew-normal random vectors. It extends the chi-square properties related to univariate and multivariate skew-normal distributions.

Keywords: Chi-square distribution; Invariance; Generalized skew-normal distribution

1. Introduction

The family of skew-normal distributions (Azzalini, 1985; Azzalini and Dalla Valle, 1996) is a flexible and tractable parametric model where the skewness can be governed by one or a few parameters. One nice property of the skew-normal distribution is that it has a close relationship with chi-square distributions. Genton and Loperfido (2002) proved a distributional invariance property of even functions in generalized skew-elliptical (GSE) random vectors. In this note, we establish an equivalence between chi-square distributions and generalized skew-normal distributions, a special case of GSE where the elliptical component is a normal probability density function. We first present the distributional invariance property for GSE random vectors.

Proposition 1 (Invariance). *Let* $\mathbf{X}$ *be a p-dimensional random vector following a GSE distribution with probability density function*

$$g(\mathbf{x}) = 2f(\mathbf{x})\pi(\mathbf{x}), \tag{1}$$

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where $f(\mathbf{x})$ *is the probability density function of an elliptical random vector, and* $\pi(\mathbf{x}): \mathbb{R}^p \to [0,1]$ *is a skewing function, i.e.* $0 \leqslant \pi(\mathbf{x}) \leqslant 1$ *and* $\pi(-\mathbf{x}) = 1 - \pi(\mathbf{x})$. *Then, the distribution of an even function* $\tau(\mathbf{X})$, *i.e.* $\tau(-\mathbf{x}) = \tau(\mathbf{x})$, *does not depend on the skewing function* $\pi(\mathbf{x})$.

Proof. Consider the characteristic function $h(t) = E[\mathrm{e}^{\mathrm{i}\tau(\mathbf{X})t}]$ of $\tau(\mathbf{X})$. We have

$$\begin{aligned} h(t) &= \int_{\mathbb{R}^p} 2\mathrm{e}^{\mathrm{i}\tau(\mathbf{x})t} f(\mathbf{x})\pi(\mathbf{x})\,\mathrm{d}\mathbf{x}, \\ &= \int_{A^+} 2\mathrm{e}^{\mathrm{i}\tau(\mathbf{x})t} f(\mathbf{x})\pi(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{A^-} 2\mathrm{e}^{\mathrm{i}\tau(\mathbf{x})t} f(\mathbf{x})\pi(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= 2\int_{A^+} \mathrm{e}^{\mathrm{i}\tau(\mathbf{x})t} f(\mathbf{x})[\pi(\mathbf{x}) + 1 - \pi(\mathbf{x})]\,\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^p} \mathrm{e}^{\mathrm{i}\tau(\mathbf{x})t} f(\mathbf{x})\,\mathrm{d}\mathbf{x}, \end{aligned}$$

does not depend on $\pi(\mathbf{x})$, where $A^+ = \{(x_1, \ldots, x_p), x_1 \geqslant 0\}$ and $A^- = \{(x_1, \ldots, x_p), x_1 < 0\}$. □

This property shows that even functions of two random vectors with two GSE distributions are identically distributed, if the elliptical components in their probability density functions (1) are the same. It is interesting to investigate the opposite: if even functions of two random vectors with GSE distributions are identically distributed, do they have the same elliptical components? In the univariate case $p = 1$, we have the following result.

Proposition 2 (Invariance: univariate case). *If* $X \sim 2f(x)\pi(x)$ *and* $\tilde{X} \sim 2\tilde{f}(x)\tilde{\pi}(x)$, *where* $2f(x)\pi(x)$ *and* $2\tilde{f}(x)\tilde{\pi}(x)$ *are two GSE probability density functions, then*

$$\begin{aligned} f(x) = \tilde{f}(x) &\Leftrightarrow \tau(X) \stackrel{\mathrm{d}}{=} \tau(\tilde{X}), \quad \forall \text{ even function } \tau \\ &\Leftrightarrow X^2 \stackrel{\mathrm{d}}{=} \tilde{X}^2. \end{aligned}$$

Proof. By Proposition 1, we have

$$f(x) = \tilde{f}(x) \Rightarrow \tau(X) \stackrel{\mathrm{d}}{=} \tau(\tilde{X}), \quad \forall \text{ even } \tau.$$

Note that since x^2 is an even function, it is trivial that

$$\tau(X) \stackrel{\mathrm{d}}{=} \tau(\tilde{X}), \quad \forall \text{ even } \tau \Rightarrow X^2 \stackrel{\mathrm{d}}{=} \tilde{X}^2.$$

Now, we only need to show

$$X^2 \stackrel{\mathrm{d}}{=} \tilde{X}^2 \Rightarrow f(x) = \tilde{f}(x).$$

Let X^s and $\tilde{X}^s$ be two random variables with probability density functions $f(x)$ and $\tilde{f}(x)$, respectively. Then by Proposition 1, for $\forall x \geqslant 0$:

$$P(X^2 \leqslant x) = P((X^s)^2 \leqslant x) = \int_{-\sqrt{x}}^{\sqrt{x}} f(t)\,\mathrm{d}t = 2\int_0^{\sqrt{x}} f(t)\,\mathrm{d}t,$$

$$P(\tilde{X}^2 \leqslant x) = P((\tilde{X}^s)^2 \leqslant x) = \int_{-\sqrt{x}}^{\sqrt{x}} \tilde{f}(t)\,\mathrm{d}t = 2\int_0^{\sqrt{x}} \tilde{f}(t)\,\mathrm{d}t.$$

Because $X^2 \overset{\mathrm{d}}{=} \tilde{X}^2$, we have $P(X^2 \leqslant x) = P(\tilde{X}^2 \leqslant x)$, or

$$\int_0^{\sqrt{x}} f(t)\,\mathrm{d}t = \int_0^{\sqrt{x}} \tilde{f}(t)\,\mathrm{d}t.$$

Note that since $f(x)$ and $\tilde{f}(x)$ are symmetric about 0, we get $f(x) = \tilde{f}(x)$, $\forall x \in (-\infty, \infty)$. $\square$

2. Equivalence

As a special case of Proposition 2, we have the following equivalence.

Corollary 1 (Chi-square characterization: univariate case). *Let $g(x)$ be the probability density function of a random variable X. Then $X^2 \sim \chi_1^2$ if and only if there exists a skewing function $\pi(x)$ such that $g(x) = 2\phi_1(x)\pi(x)$, where $\phi_1(x) = (1/\sqrt{2\pi})\exp(-x^2/2)$ is the probability density function of the standard normal distribution.*

Corollary 1 characterizes the distribution of all random variables whose squares have a χ_1^2 distribution. A similar result holds for multivariate distributions.

Corollary 2 (Chi-square characterization: multivariate case). *Let $\mathbf{X}$ be a p-dimensional random vector and $\mathbf{X} \sim g(\mathbf{x}) = 2f(\mathbf{x})\pi(\mathbf{x})$, where $f(\mathbf{x})$ is spherical, i.e. $f(\mathbf{x}) = \psi(\mathbf{x}^{\mathrm{T}}\mathbf{x})$. Then*

$$\mathbf{X}^{\mathrm{T}}\mathbf{X} \sim \chi_p^2 \quad \Leftrightarrow \quad f(\mathbf{x}) = \phi_p(\mathbf{x}),$$

where $\phi_p(\mathbf{x}) = (2\pi)^{-p/2}\exp(-\mathbf{x}^{\mathrm{T}}\mathbf{x}/2)$ is the probability density function of a standard normal random vector.

Proof. Let $\mathbf{Y}$ be a random vector with density $f(\mathbf{y})$. By Proposition 1, $\mathbf{X}^{\mathrm{T}}\mathbf{X} \sim \chi_p^2$ if and only if $\mathbf{Y}^{\mathrm{T}}\mathbf{Y} \sim \chi_p^2$. Therefore, by solving

$$\int_{\mathbf{y}^{\mathrm{T}}\mathbf{y} \leqslant u} \psi(\mathbf{y}^{\mathrm{T}}\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_0^u \frac{1}{\Gamma(p/2)} \exp\left(-\frac{t}{2}\right) t^{p/2-1}\,\mathrm{d}t,$$

we obtain $\psi(u) = (2\pi)^{-p/2}\exp(-u/2)$. Hence, we have

$$f(\mathbf{x}) = \psi(\mathbf{x}^{\mathrm{T}}\mathbf{x}) = (2\pi)^{-p/2}\exp\left(-\frac{\mathbf{x}^{\mathrm{T}}\mathbf{x}}{2}\right) = \phi_p(\mathbf{x}). \qquad \square$$

Note that the sphericity assumption of $f(\mathbf{x})$ is important. Otherwise $\mathbf{X}^{\mathrm{T}}\mathbf{X} \sim \chi_p^2$ may not imply $f(\mathbf{x}) = \phi_p(\mathbf{x})$ as shown in the following example. Consider a bivariate random vector (X, Y) with probability density function:

$$f(x, y) = \begin{cases} (1+\varepsilon)\phi_2(x, y) & \text{if } xy > 0, \\ (1-\varepsilon)\phi_2(x, y) & \text{if } xy < 0, \end{cases}$$

where $0 < \varepsilon < 1$ is a constant. It is easy to show that $f(x, y)$ is symmetric about $(0,0)$ and $X^2 + Y^2 \sim \chi_2^2$, but $f(x, y) \neq \phi_2(x, y)$.

References

Azzalini, A., 1985. A class of distributions which includes the normal ones. Scand. J. Statist. 12, 171–178.

Azzalini, A., Dalla Valle, A., 1996. The multivariate skew-normal distribution. Biometrika 83, 715–726.

Genton, M.G., Loperfido, N., 2002. Generalized skew-elliptical distributions and their quadratic forms. Institute of Statistics Mimeo Series #2539.