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The multivariate skew-slash distribution

Jing Wang, Marc G. Genton*

Department of Statistics, North Carolina State University, Box 8203, Raleigh NC 27695-8203, USA

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Abstract

The slash distribution is often used as a challenging distribution for a statistical procedure. In this article, we define a skewed version of the slash distribution in the multivariate setting and derive several of its properties. The multivariate skew-slash distribution is shown to be easy to simulate from and can therefore be used in simulation studies. We provide various examples for illustration.

MSC: 60E05; 62H05

Keywords: Cauchy; Configural polysampling; Kurtosis; Multivariate distributions; Skewness; Skew-normal; Slash

1. Introduction

Despite the central role played by the *magic bell-shaped* normal distribution in statistics, there has been a sustained interest among statisticians in constructing more challenging distributions for their procedures. Indeed, if one can perform satisfactorily under two extreme scenarios, it is commonly believed that the performance will be reasonably good under intermediate situations. This idea is the cornerstone of configural polysampling, a small sample approach to robustness described by Morgenthaler and Tukey (1991). A first family of scenarios can be represented by a finite mixture of normal distributions. However, this family does not contain any real challenge since its members always have a finite variance. A second more popular class in this scenario is the t_ν distribution with ν degrees of freedom,

* Corresponding author. Tel.: +1-919-515-1936; fax: +1-919-515-1169.
E-mail addresses: jwang6@stat.ncsu.edu (J. Wang), genton@stat.ncsu.edu (M.G. Genton).

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which converges to the normal distribution for $v \to \infty$. The other extreme given by $v = 1$ is a Cauchy distribution, that is, the distribution of the ratio between two independent standard normal random variables. This distribution has heavier tails than the normal distribution and does not possess finite moments or cumulants. However, its sharp central peak has often been considered as unrealistic in representing real data.

Another family of scenarios is described by the standard slash distribution, representing the distribution of the ratio $X = Z/(U^{1/q})$ of a standard normal random variable Z to an independent uniform random variable U on the interval $(0, 1)$ raised to the power $1/q$, $q > 0$. When $q = 1$ we obtain the canonical slash, whereas $q \to \infty$ yields the normal distribution. The probability density function (pdf) of the univariate slash distribution is symmetric about the origin and has heavier tails than those of the normal density, with, for the canonical slash, the same tail heaviness as the Cauchy. However, it is less peaked in the center and thus more realistic in representing data. Effectively, straightforward algebra yields the pdf

$$\psi(x; q) = q \int_0^1 u^q \phi(xu)\,\mathrm{d}u$$

and the cumulative distribution function (cdf)

$$\Psi(x; q) = q \int_0^1 u^{q-1} \Phi(xu)\,\mathrm{d}u, \tag{1}$$

where ϕ and Φ denote the standard normal pdf and cdf, respectively. In particular, we have

$$\psi(0; q) = (q/(q+1))\phi(0) = q/(\sqrt{2\pi}(q+1))$$

and $\Psi(0; q) = \frac{1}{2}$. Moreover, closed-form expressions for the pdf can be computed, for instance

$$\psi(x; 1) = \begin{cases} (\phi(0) - \phi(x))/x^2, & x \neq 0, \\ \phi(0)/2, & x = 0 \end{cases}$$

and

$$\psi(x; 2) = \begin{cases} 2((\Phi(x) - \Phi(0))/x - \phi(x))/x^2, & x \neq 0, \\ 2\phi(0)/3, & x = 0. \end{cases}$$

The expectation and the variance of the standard slash distribution are given by $\mathrm{E}(X) = 0$ for $q > 1$ and $\mathrm{Var}(X) = q/(q-2)$ for $q > 2$. A general slash distribution is obtained by scale multiplication and location shift of a standard slash random variable, see Rogers and Tukey (1972), and Mosteller and Tukey (1977) for further properties. Kafadar (1982) discussed the maximum likelihood estimation of the location and scale parameters of this family. The slash distribution has been mainly used in simulation studies because it represents an extreme situation, see for example Andrews et al. (1972), Gross (1973), and Morgenthaler and Tukey (1991). In this paper, we introduce an additional challenge by defining a skewed version of the slash distribution.

Another approach to introduce challenges for statistical procedures is to define skewed distributions. A simple departure from the normal distribution has been proposed by Azzalini (1985) who defined the skew-normal distribution with pdf

$$2\phi(x; \mu, \sigma^2)\Phi(\alpha(x - \mu)), \tag{2}$$

where $\phi(x;\mu,\sigma^2)$ is the pdf of a normal random variable with mean μ and variance σ^2, and α is a shape parameter controlling skewness. For $\alpha=0$, the pdf (2) reduces to the normal one, whereas for $\alpha>0$ or $\alpha<0$ the pdf is skewed to the right or the left, respectively. An extension of (2) to the multivariate setting was proposed by Azzalini and Dalla Valle (1996), defining the pdf

$$2\phi_p(\boldsymbol{x};\boldsymbol{\mu},\Sigma)\Phi(\boldsymbol{\alpha}^{\mathrm{T}}(\boldsymbol{x}-\boldsymbol{\mu})),\quad \boldsymbol{x}\in\mathbb{R}^p, \tag{3}$$

where $\phi_p(\boldsymbol{x};\boldsymbol{\mu},\Sigma)$ is the p-dimensional normal pdf with mean $\boldsymbol{\mu}$ and correlation matrix Σ, $\Phi(\cdot)$ is the standard normal cdf $\mathrm{N}(0,1)$, and $\boldsymbol{\alpha}$ is a p-dimensional shape parameter. A p-dimensional random vector $\boldsymbol{X}$ with a multivariate skew-normal distribution is denoted by $\boldsymbol{X}\sim \mathrm{SN}_p(\boldsymbol{\mu},\Sigma,\boldsymbol{\alpha})$. Its expectation and variance are given by

$$\mathrm{E}(\boldsymbol{X})=\boldsymbol{\mu}+\sqrt{\frac{2}{\pi}}\boldsymbol{\delta}, \tag{4}$$

$$\mathrm{Var}(\boldsymbol{X})=\Sigma-\frac{2}{\pi}\boldsymbol{\delta}\boldsymbol{\delta}^{\mathrm{T}}, \tag{5}$$

where $\boldsymbol{\delta}=\Sigma\boldsymbol{\alpha}/\sqrt{1+\boldsymbol{\alpha}^{\mathrm{T}}\Sigma\boldsymbol{\alpha}}$, see e.g. Genton et al. (2001).

It is now natural to construct univariate and multivariate distributions that combine skewness with heavy tails. For instance, one can define skew-t distributions (Branco and Dey, 2001; Jones and Faddy, 2003; Sahu et al., 2003), skew-Cauchy distributions (Arnold and Beaver, 2000), skew-elliptical distributions (Azzalini and Capitanio, 1999; Branco and Dey, 2001; Sahu et al., 2003; Genton and Loperfido, 2005), or other skew-symmetric distributions (Wang et al., 2004). In this article, we define a multivariate skew-slash distribution and study its properties and applications.

This article is organized as follows. In Section 2, we define the multivariate slash distribution and derive various of its properties. For example, we show that the slash distribution is invariant under linear transformations and that its moments are analytically tractable. In Section 3, we define the multivariate skew-slash distribution and provide several examples revealing its skewness and tail behavior. The results of a small simulation study are reported in Section 4 along with two illustrative applications. We conclude in Section 5.

2. The multivariate slash distribution

In this section, we define a multivariate slash distribution and derive its pdf. We show that the multivariate slash distribution is invariant under linear transformations and derive its moments. In the sequel, we denote the p-dimensional multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ by $\mathrm{N}_p(\boldsymbol{\mu},\Sigma)$, its pdf by $\phi_p(\boldsymbol{x};\boldsymbol{\mu},\Sigma)$, and the standard uniform distribution on the interval $(0,1)$ by $\mathrm{U}(0,1)$.

Definition 1. A random vector $\boldsymbol{X}\in\mathbb{R}^p$ has a p-dimensional slash distribution with location parameter $\boldsymbol{\mu}$, positive definite scale matrix parameter Σ, and tail parameter $q>0$, denoted by $\boldsymbol{X} \sim \mathrm{SL}_p(\boldsymbol{\mu}, \Sigma, q)$, if

$$\boldsymbol{X} = \Sigma^{1/2}\frac{\boldsymbol{Y}}{U^{1/q}} + \boldsymbol{\mu}, \tag{6}$$

where $\boldsymbol{Y} \sim \mathrm{N}_p(\boldsymbol{0}, I_p)$ is independent of $U \sim \mathrm{U}(0, 1)$.

When $\boldsymbol{\mu} = \boldsymbol{0}$ and $\Sigma = I_p$, $\boldsymbol{X}$ in (6) has a standard multivariate slash distribution. The pdf of the random vector $\boldsymbol{X}$ in (6) is easily shown to be

$$\begin{aligned}\psi_p(\boldsymbol{x}; \boldsymbol{\mu}, \Sigma, q) &= q\int_0^1 u^{q+p-1}\phi_p(u\boldsymbol{x}; u\boldsymbol{\mu}, \Sigma)\,\mathrm{d}u \\ &= \begin{cases} \frac{q2^{(q+p)/2-1}\gamma((q+p)/2; \|\Sigma^{-1/2}(\boldsymbol{x}-\boldsymbol{\mu})\|^2/2)}{(2\pi)^{p/2}\|\Sigma^{-1/2}(\boldsymbol{x}-\boldsymbol{\mu})\|^{q+p}}, & \boldsymbol{x} \neq \boldsymbol{0}, \\ \frac{q}{q+p}\left(\frac{1}{2\pi}\right)^{p/2}, & \boldsymbol{x} = \boldsymbol{0}, \end{cases}\end{aligned}$$

where $\gamma(a; z) = \int_0^z t^{a-1}\mathrm{e}^{-t}\,\mathrm{d}t = \sum_{k=0}^{\infty}\frac{(-1)^k z^{a+k}}{k!(a+k)}$, and $\|\Sigma^{-1/2}(\boldsymbol{x}-\boldsymbol{\mu})\| = \sqrt{(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$. Note that the standard slash random vector in (6) is a scale mixture of the normal model (see e.g. Fang et al., 1990) and so it can be represented as:

$$\boldsymbol{X}|(U = u) \sim \mathrm{N}_p(\boldsymbol{0}, u^{-1/q}I_p) \quad \text{with } U \sim \mathrm{U}(0, 1).$$

Next consider the linear transformation $\boldsymbol{V} = \boldsymbol{b} + A\boldsymbol{X}$, where $\boldsymbol{X}$ has the multivariate slash distribution $\boldsymbol{X} \sim \mathrm{SL}_p(\boldsymbol{\mu}, \Sigma, q)$, $\boldsymbol{b}$ is a vector in $\mathbb{R}^p$, and A is a nonsingular matrix. The Jacobian determinant of the transformation is $|A|^{-1}$ and hence the pdf of $\boldsymbol{V}$ is $|A|^{-1}\psi_p(A^{-1}(\boldsymbol{v} - \boldsymbol{b}); \boldsymbol{\mu}, \Sigma, q)$, showing that $\boldsymbol{V}$ has a multivariate slash distribution $\mathrm{SL}_p(\boldsymbol{b} + A\boldsymbol{\mu}, A\Sigma A^{\mathrm{T}}, q)$. This implies that the slash distribution is invariant under linear transformations and this is summarized in the following proposition.

Proposition 1. *If* $\boldsymbol{X} \sim \mathrm{SL}_p(\boldsymbol{\mu}, \Sigma, q)$, *then its linear transformation* $\boldsymbol{V} = \boldsymbol{b} + A\boldsymbol{X} \sim \mathrm{SL}_p(\boldsymbol{b} + A\boldsymbol{\mu}, A\Sigma A^{\mathrm{T}}, q)$.

It can be checked that Proposition 1 still holds when A is only a full row rank matrix and therefore marginal distributions of $\boldsymbol{X} \sim \mathrm{SL}_p(\boldsymbol{\mu}, \Sigma, q)$ are still of the slash type. Alternatively, this follows from the fact that marginal distributions of $\boldsymbol{Y} \sim \mathrm{N}_p(\boldsymbol{0}, I_p)$ in (6) are still normal.

Now we consider the moments of $\boldsymbol{X}$. The moments of a uniform random variable $U \sim \mathrm{U}(0, 1)$ are given by

$$\mathrm{E}(U^{-k/q}) = \frac{q}{q-k}, \quad q > k. \tag{7}$$

Because $\boldsymbol{Y}$ and U are independent in (6), the moments of $\boldsymbol{X}$ follow immediately from the moments of $\boldsymbol{Y}$ and (7). For instance, the first two moments of $\boldsymbol{X}$ for the multivariate slash distribution are shown in the following proposition.

Proposition 2. *If* $\boldsymbol{X} \sim \mathrm{SL}_p(\boldsymbol{\mu}, \Sigma, q)$, *then the expectation and variance of* $\boldsymbol{X}$ *are given by*

$$\mathrm{E}(\boldsymbol{X}) = \boldsymbol{\mu} \quad \text{if } q > 1,$$
$$\mathrm{Var}(\boldsymbol{X}) = \frac{q}{q-2}\Sigma \quad \text{if } q > 2.$$

3. The multivariate skew-slash distribution

In this section, we define a multivariate skew-slash distribution and derive its pdf. We also provide another definition based on a stochastic representation of the skew-slash distribution, which is useful for simulations. Both definitions lead to the same probability density function. We show that the multivariate skew-slash distribution is invariant under linear transformations.

Definition 2. A random vector $\boldsymbol{X} \in \mathbb{R}^p$ has a p-dimensional skew-slash distribution with location parameter $\boldsymbol{\mu}$, positive definite scale matrix parameter Σ, tail parameter $q > 0$, and skewness parameter $\boldsymbol{\alpha}$, denoted by $\boldsymbol{X} \sim \mathrm{SSL}_p(\boldsymbol{\mu}, \Sigma, q, \boldsymbol{\alpha})$, if

$$\boldsymbol{X} = \Sigma^{1/2}\frac{\boldsymbol{Y}}{U^{1/q}} + \boldsymbol{\mu}, \tag{8}$$

where $\boldsymbol{Y} \sim \mathrm{SN}_p(\boldsymbol{0}, I_p, \boldsymbol{\alpha})$ is independent of $U \sim \mathrm{U}(0, 1)$.

When $\boldsymbol{\mu} = \boldsymbol{0}$ and $\Sigma = I_p$, $\boldsymbol{X}$ in (8) has a standard multivariate skew-slash distribution. The pdf of the random vector $\boldsymbol{X}$ in (8) is then

$$\eta_p(\boldsymbol{x}; \boldsymbol{\mu}, \Sigma, q, \boldsymbol{\alpha}) = 2q\int_0^1 u^{q+p-1}\phi_p(u\boldsymbol{x}; u\boldsymbol{\mu}, \Sigma)\Phi(u\boldsymbol{\alpha}^{\mathrm{T}}\Sigma^{-1/2}(\boldsymbol{x}-\boldsymbol{\mu}))\,\mathrm{d}u. \tag{9}$$

In the univariate case, i.e. for $p=1$, the distribution of $T=|X|$ has density $2\psi(t; q)I(t>0)$ in both cases when $X \sim \mathrm{SL}(0, 1, q)$ and when $X \sim \mathrm{SSL}(0, 1, q, \alpha)$. This invariance property holds also for the distribution of $\|\boldsymbol{X}\|$ in the multivariate case, which follows directly from the stochastic representation in (8) and considering the well known result about the invariance of the distribution of $\|\boldsymbol{Y}\|$, $\boldsymbol{Y} \sim \mathrm{SN}_p(\boldsymbol{0}, I_p, \boldsymbol{\alpha})$, with respect to $\boldsymbol{\alpha}$. This invariance property is similar to the one derived by Genton et al. (2001) for the skew-normal distribution, Genton and Loperfido (2005) for generalized skew-elliptical distributions, and by Wang et al. (2004) for skew-symmetric distributions.

Next consider the linear transformation $\boldsymbol{V} = \boldsymbol{b} + A\boldsymbol{X}$, where $\boldsymbol{X}$ has the multivariate skew-slash distribution $\boldsymbol{X} \sim \mathrm{SSL}_p(\boldsymbol{\mu}, \Sigma, q, \boldsymbol{\alpha})$, $\boldsymbol{b}$ is a vector in $\mathbb{R}^p$, and A is a nonsingular matrix. The Jacobian determinant of the transformation is $|A|^{-1}$ and hence the pdf of $\boldsymbol{V}$ is $|A|^{-1}\eta_p(A^{-1}(\boldsymbol{v}-\boldsymbol{b}); \boldsymbol{\mu}, \Sigma, q, \boldsymbol{\alpha})$, showing that $\boldsymbol{V}$ has a multivariate skew-slash distribution $\mathrm{SSL}_p(\boldsymbol{b} + A\boldsymbol{\mu}, A\Sigma A^{\mathrm{T}}, q, A^{-\mathrm{T}}\boldsymbol{\alpha})$. This implies that the skew-slash distribution is invariant under linear transformations and is summarized in the following proposition.

Proposition 3. *If* $\boldsymbol{X} \sim \mathrm{SSL}_p(\boldsymbol{\mu}, \Sigma, q, \boldsymbol{\alpha})$, *then its linear transformation* $\boldsymbol{V} = \boldsymbol{b} + A\boldsymbol{X} \sim \mathrm{SSL}_p(\boldsymbol{b} + A\boldsymbol{\mu}, A\Sigma A^{\mathrm{T}}, q, A^{-\mathrm{T}}\boldsymbol{\alpha})$.

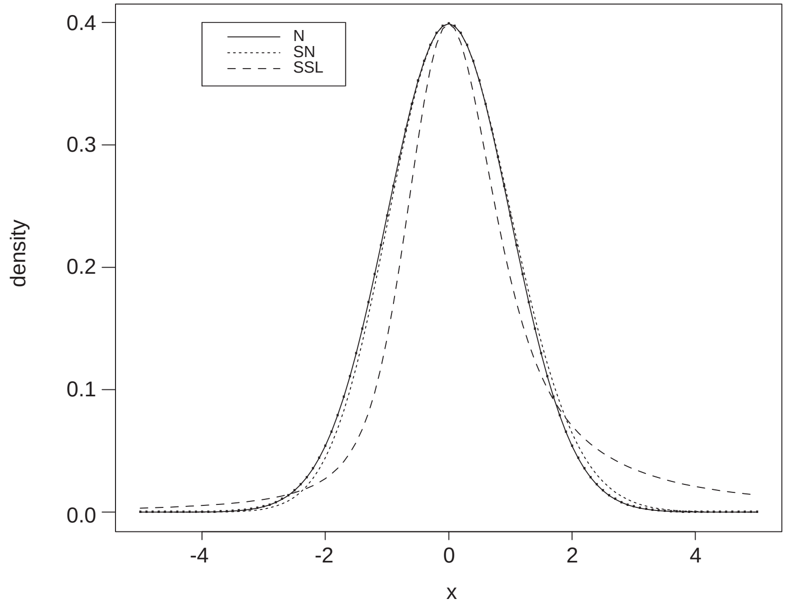

Fig. 1. Density curves of the univariate normal, skew-normal, and skew-slash distributions.

Here again, it can be checked that Proposition 3 still holds when A is only a full row rank matrix and therefore marginal distributions of $\boldsymbol{X} \sim \mathrm{SSL}_p(\boldsymbol{\mu}, \Sigma, q, \boldsymbol{\alpha})$ are still of the skew-slash type. Alternatively, this follows from the fact that marginal distributions of $\boldsymbol{Y} \sim \mathrm{SN}_p(\mathbf{0}, I_p, \boldsymbol{\alpha})$ in (8) are still skew-normal, see Azzalini and Dalla Valle (1996).

The multivariate skew-slash distribution $\mathrm{SSL}_p(\boldsymbol{\mu}, \Sigma, q, \boldsymbol{\alpha})$ reduces to the skew-normal distribution $\mathrm{SN}_p(\boldsymbol{\mu}, \Sigma, \boldsymbol{\alpha})$ when $q \to \infty$, to the slash distribution $\mathrm{SL}_p(\boldsymbol{\mu}, \Sigma, q)$ when $\boldsymbol{\alpha} = \mathbf{0}$, and to the normal distribution $\mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$ when both $\boldsymbol{\alpha} = \mathbf{0}$ and $q \to \infty$. Thus, the skew-slash distribution includes a wide variety of contour shapes. To illustrate the skewness and tail behavior of the skew-slash, we draw the density of the univariate skew-slash distribution $\mathrm{SSL}_1(0, 1, 1, 1)$ together with the densities of the standard normal distribution $\mathrm{N}_1(0, 1)$ and skew-normal distribution $\mathrm{SN}_1(0, 1, 1)$. Fig. 1 depicts the three density curves (the skew-normal and skew-slash have been centered at zero and rescaled). Note that the densities of the skew-normal and skew-slash distributions are positively skewed and that the skew-slash distribution has much heavier tails than the normal and skew-normal distribution. Actually, this skew-slash distribution does not have finite mean and variance, see Proposition 4. Fig. 2 depicts contour plots of the standard bivariate skew-slash pdf for the parameter values $q = 5$, $\boldsymbol{\alpha} = (1, 1)^{\mathrm{T}}$ (left panel) and $q = 1$, $\boldsymbol{\alpha} = (0.5, 0.2)^{\mathrm{T}}$ (right panel). The right panel shows a case with heavier tails than the left panel, but both exhibit skewness.

Next we discuss an equivalent definition of the multivariate skew-slash distribution based on an approach described by Arnold and Beaver (2002). For simplicity of the exposition, we set $\boldsymbol{\mu} = \mathbf{0}$ and $\Sigma = I_p$. Consider the conditional distribution of $\boldsymbol{W}$ given $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{W} > W_0$, where $\boldsymbol{\alpha} \in \mathbb{R}^p$, and $(W_0, \boldsymbol{W}^{\mathrm{T}}) \sim \mathrm{SL}_{p+1}(\mathbf{0}, I_{p+1}, q)$ with $\boldsymbol{W} = (W_1, W_2, \ldots, W_p)^{\mathrm{T}}$. The

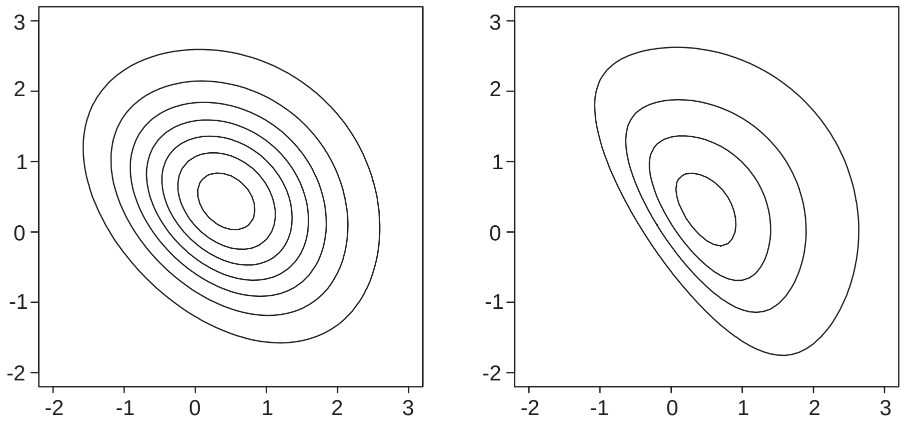

Fig. 2. Contour plots of the standard bivariate skew-slash pdf for $q = 5$, $\boldsymbol{\alpha} = (1, 1)^{\mathrm{T}}$ (left panel) and $q = 1$, $\boldsymbol{\alpha} = (0.5, 0.2)^{\mathrm{T}}$ (right panel).

joint probability density function of $(W_0, \boldsymbol{W})$ conditional on $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{W} > W_0$ is given by

$$\frac{\psi_{p+1}(w_0, \boldsymbol{w}; \mathbf{0}, I_{p+1}, q)\mathbf{1}(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{w} > w_0)}{P\{W_0 - \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{W} < 0\}}, \quad w_0 \in \mathbb{R}, \quad \boldsymbol{w} \in \mathbb{R}^p.$$

It follows from the symmetry of the slash distribution that $P\{W_0 - \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{W} < 0\} = \frac{1}{2}$. Integrating out w_0 yields the density of $\boldsymbol{W}$ which is given by

$$\begin{aligned}
&\int_{-\infty}^{\infty} 2\psi_{p+1}(w_0, \boldsymbol{w}; \mathbf{0}, I_{p+1}, q)\mathbf{1}(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{w} > w_0)\,\mathrm{d}w_0 \\
&\quad = 2q(2\pi)^{-(p+1)/2} \int_{-\infty}^{\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{w}} \int_0^1 u^{p+q} \\
&\qquad \times \exp(-u^2 w_0^2/2) \exp\left(-u^2 \sum_{i=1}^{p} w_i^2/2\right) \mathrm{d}u\,\mathrm{d}w_0 \\
&\quad = 2q \int_0^1 u^{p+q-1} \phi_p(u\boldsymbol{w}; \mathbf{0}, I_p) \Phi(u\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{w})\,\mathrm{d}u, \quad \boldsymbol{w} \in \mathbb{R}^p,
\end{aligned}$$

which is exactly the density $\eta_p(\boldsymbol{w}; \mathbf{0}, I_p, q, \boldsymbol{\alpha})$ in (9) of the multivariate skew-slash distribution $\mathrm{SSL}_p(\mathbf{0}, I_p, q, \boldsymbol{\alpha})$. This result proves the equivalence between Definition 2 and the definition based on conditioning. Consequently either of these two definitions can be used to simulate from the multivariate skew-slash distribution, see Section 4.1.

We now compute the moments of the skew-slash distribution. Using the same argument as for the multivariate slash distribution, the moments of $\boldsymbol{X} \sim \mathrm{SSL}_p(\boldsymbol{\mu}, \Sigma, q, \boldsymbol{\alpha})$ follow readily from (7) and the moments of $\boldsymbol{Y} \sim \mathrm{SN}_p(\mathbf{0}, I_p, \boldsymbol{\alpha})$, see e.g. Genton et al. (2001). For instance, the first two moments of $\boldsymbol{X}$ for the multivariate skew-slash distribution are shown in the following proposition.

Proposition 4. *If* $\boldsymbol{X} \sim \mathrm{SSL}_p(\boldsymbol{\mu}, \Sigma, q, \boldsymbol{\alpha})$, *then the expectation and variance of* $\boldsymbol{X}$ *are given by*

$$\mathrm{E}(\boldsymbol{X}) = \boldsymbol{\mu} + \frac{q}{q-1}\sqrt{\frac{2}{\pi}}\boldsymbol{\delta} \quad \text{if } q > 1,$$

$$\mathrm{Var}(\boldsymbol{X}) = \frac{q}{q-2}\Sigma - \frac{2}{\pi}\left(\frac{q}{q-1}\right)^2 \boldsymbol{\delta}\boldsymbol{\delta}^{\mathrm{T}} \quad \text{if } q > 2,$$

where $\boldsymbol{\delta} = \Sigma\boldsymbol{\alpha}/\sqrt{1+\boldsymbol{\alpha}^{\mathrm{T}}\Sigma\boldsymbol{\alpha}}$.

Note that when there is no skewness, i.e. $\boldsymbol{\alpha} = \boldsymbol{\delta} = \mathbf{0}$, the expectation and variance in Proposition 4 reduce to the expressions given in Proposition 2, whereas when $q \to \infty$ (no heavy tail behavior) they reduce to the expressions given by (4) and (5).

4. Applications

In this section, we present three applications of the skew-slash distribution. The first one illustrates the use of the skew-slash distribution in simulation studies, whereas the other two involve the statistical analysis of real data sets.

4.1. Skew-slash distributions in simulation studies

The skew-slash distribution can be used in simulation studies as a challenging distribution for a statistical procedure. As an illustration, we perform a small simulation to study the behavior of two location estimators, the sample mean and the sample median, in four different univariate settings. We consider two symmetric distributions, a standard normal $\mathrm{N}_1(0, 1)$ and a slash $\mathrm{SL}_1(0, 1, 2)$, and two asymmetric distributions, a skew-normal $\mathrm{SN}_1(0, 1, 3)$ and a skew-slash $\mathrm{SSL}_1(0, 1, 2, 3)$. The location means of the asymmetric distributions are adjusted to zero, so that all four distributions are comparable. Thus, this setting represents four distributions with same mean, but with different tail behavior and skewness. Actually, with $q = 2$, the variance of the slash and skew-slash distributions is infinite. We simulate 500 samples of size $n = 100$ from each of these four distributions. On each sample, we compute the sample mean and sample median and report the boxplot for each distribution in Fig. 3. In the left panel, we observe that all boxplots of the estimated means are centered around zero but have larger variability for the heavy tailed distributions (the slash and the skew-slash). In the right panel, we see that the boxplot of the estimated medians has a slightly larger variability than the boxplot for the estimated means at the normal distribution, but has a much smaller variability at the slash distribution. This indicates that the median is a robust estimator of location at symmetric distributions. On the other hand, the median estimator becomes biased as soon as unexpected skewness arises in the underlying distribution, see the boxplot of the estimated medians of the skew-normal distributions. This effect is even more severe under both skewness and heavy-tails represented by the skew-slash distribution. Although this simulation is very simple, it illustrates the use of non-normal distributions to challenge the behavior of statistical procedures.

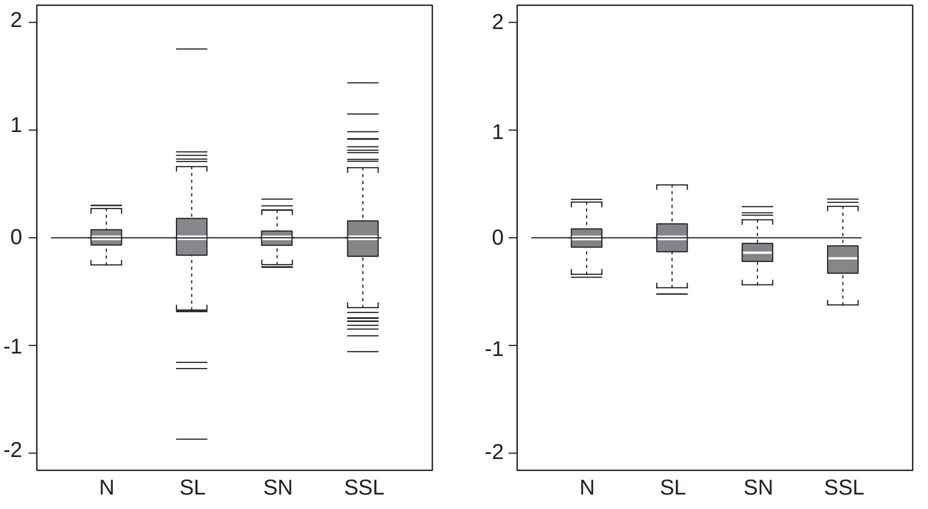

Fig. 3. Boxplots of the sample mean (left panel) and sample median (right panel) on 500 samples of size $n = 100$ from four distributions: $N_1(0, 1)$; $SL_1(0, 1, 2)$; $SN_1(0, 1, 3)$; $SSL_1(0, 1, 2, 3)$ (the means of SN_1 and SSL_1 are adjusted to zero).

4.2. Fiber-glass data set

This application is concerned with a unidimensional data set of $n = 63$ breaking strengths values of 1.5 cm long glass fibers. Jones and Faddy (2003) and Azzalini and Capitanio (2003) fit two forms of skew-t distributions to these data. They both note skewness on the left as well as heavy tail behavior. We instead fit a skew-slash distribution. The fitted parameters, obtained by maximizing the likelihood function, are $\hat{\mu} = 1.81$, $\hat{\sigma} = 3.33$, $\hat{q} = 3.33$, and $\hat{\alpha} = -3.00$. The negative value of $\hat{\alpha}$ indicates skewness on the left and the small value of $\hat{q}$ indicates a heavy tail behavior. Fig. 4 depicts a histogram of the data and the fitted skew-slash probability density function, which has a finite mean and variance.

4.3. Australian athletes data set

This application deals with the Australian athletes data set analyzed by Cook and Weisberg (1994) in a normal setting and Azzalini and Dalla Valle (1996) with the skew-normal distribution. It consists of several variables measured on $n=202$ athletes and we focus on body mass index (BMI) and lean body mass (LBM). We fit a bivariate skew-slash to these data in order to investigate the possible heavier-than-normal tail behavior of (BMI, LBM). The parameters, estimated by maximizing the likelihood function, are $\hat{\boldsymbol{\mu}} = (20.24, 62.20)^{\mathrm{T}}$, $\hat{\sigma}^{1,1} = 0.38$, $\hat{\sigma}^{1,2} = -0.07$, $\hat{\sigma}^{2,2} = 0.09$, $\hat{\boldsymbol{\alpha}} = (3.95, 1.01)^{\mathrm{T}}$, and $\hat{q} = 8.98$, where $\sigma^{i,j}$ is the (i, j)-entry of the 2×2 matrix $\Sigma^{-1/2}$. The skewness parameter $\hat{\boldsymbol{\alpha}}$ indicates apparent skewness, as can be seen in Fig. 5, but the parameter $\hat{q}$ does not indicate a serious heavy-tail behavior. We use a likelihood ratio test for the null hypothesis $H_0 : q = \infty$, that is to test that a skew-normal distribution is enough. The unconstrained log likelihood function value

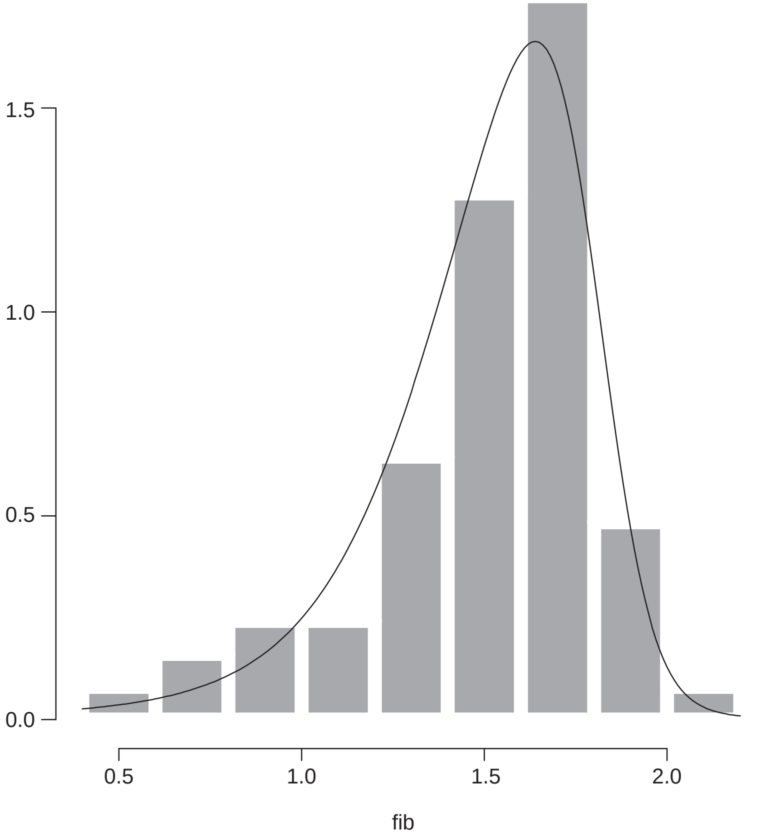

Fig. 4. Fitted skew-slash probability density function (solid line) to the fiber-glass data.

is 1213.261 and the constrained log likelihood function value is 1212.953, which yields a likelihood ratio test statistic of 0.616 and a p-value of 0.567 with an asymptotic χ_1^2 distribution under the null hypothesis. There is not enough evidence in the data to reject H_0, and therefore a skew-normal distribution would be appropriate for this example.

5. Discussion

We have introduced a multivariate skew-slash distribution, a flexible distribution that can take skewness and heavy tails into account. This distribution is useful in simulation studies where it can introduce distributional challenges in order to evaluate a statistical procedure. It is also useful in analyzing data sets that do not follow the normal law. We have used the fiber-glass data set and the Australian athletes data set for illustration. Additional flexibility can be introduced in the skew-slash distribution by allowing higher order odd polynomials in

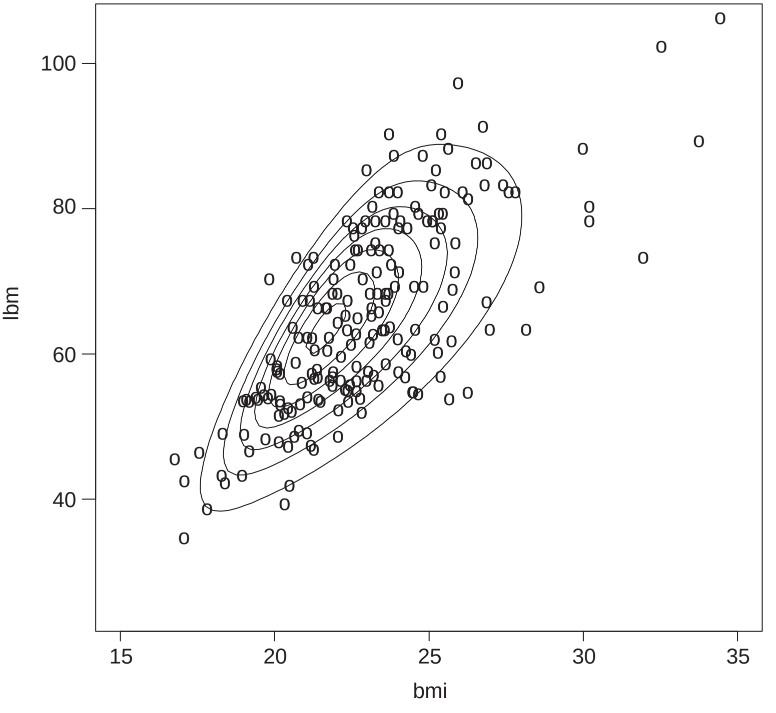

Fig. 5. Contours of the fitted bivariate skew-slash probability density function to the (BMI, LBM) variables of Australian athletes data.

the skewing function $\Phi(\cdot)$ in (9). For instance, an odd polynomial of order three would yield a distribution that can model bimodality, see Ma and Genton (2004) for further discussions on this topic.

Jones and Faddy (2003) discuss different versions of skew-t distributions. Their own proposal is developed in the univariate setting and they acknowledge that its extension to the multivariate setting (Jones, 2001) is of questionable usefulness. Families of closely related multivariate skew-t distributions are constructed by Azzalini and Capitanio (2003), and Sahu et al. (2003). However, Jones and Faddy (2003) point out that odd moments of these families are analytically intractable, which results in the unavailability of a Fisher scoring algorithm for likelihood maximization. Another family of univariate skew-t distributions is proposed by Fernández and Steel (1998). They use Bayesian fitting techniques because standard asymptotic likelihood theory is not applicable due to the discontinuity of even derivatives of their skew-t density at the origin. The multivariate skew-slash distribution introduced in this article is clearly an alternative to skew-t distributions because it can model both skewness and heavy tails. One interesting advantage of the multivariate skew-slash distribution is that its moments can be computed analytically by taking advantage of the moments of the multivariate skew-normal distribution, see the discussion in Section 3. Another attractive feature is that simulations from the multivariate skew-slash distribution are straightforward from softwares that permit simulations from the multivariate skew-normal or normal distribution.

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