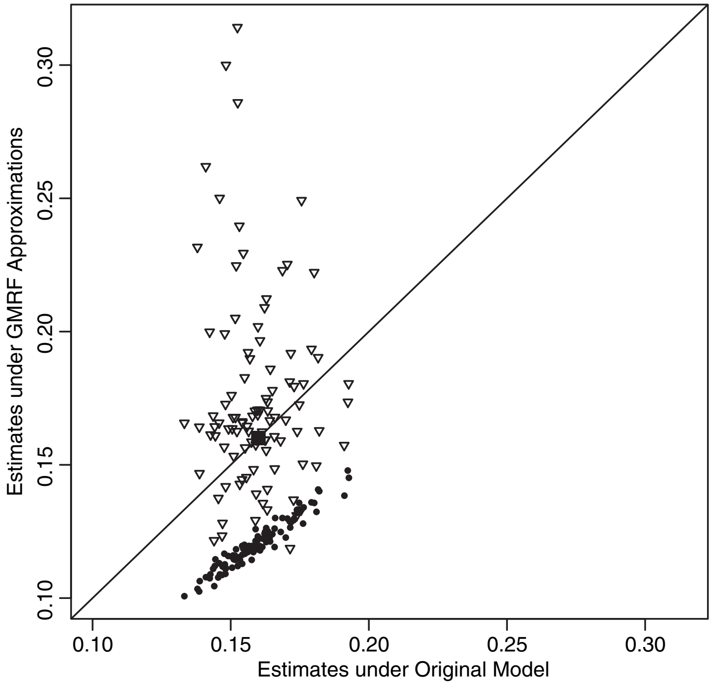

Fig. 18. Estimates for $\sigma^2\kappa^{2\nu}$ with effective range of 10 on a 20 × 20 unit grid with $\nu=1$ and $\sigma^2=2$ (the *x*-axis shows the maximum likelihood estimate, whereas the *y*-axis shows estimators maximizing two approximations to the likelihood): $\triangledown$, naive implementation with no boundary correction; ●, implementation with embedding; ■, true parameter value

the treatment of the boundary points in this context. Although many will undoubtably use these models in their non-stationary versions, non-stationarity that is an artefact of the boundary is undesirable and may introduce bias in estimating the geostatistical parameters.

We simulated 100 replications of a mean 0 Gaussian field on a 20×20 unit grid, using a Matérn covariance function with $\nu=1$ and $\sigma^2=2$, and effective range 10 ($\kappa\approx0.283$). The large effective range highlights the boundary issue, but we think that high auto-correlation is also not uncommon in geostatistical data. We fixed ν at its true value and considered three possible estimators for σ^2 and κ. The first is the maximum likelihood estimator under the original model. The second maximizes an approximate likelihood that replaces the covariance matrix $\Sigma(\sigma^2,\kappa)$ by $\sigma^2 Q(\kappa)^{-1}/\sigma^2_{\mathrm{M}}(\kappa)$, where $Q(k)$ is constructed by using the formulation from Section 2.2.1, with no special treatment of the boundary points, and $\sigma^2_{\mathrm{M}}(\kappa)$ is the marginal variance for the stochastic partial differential equation solution. The third treats the boundary points by using the embedding method suggested in Section 5.1.4 of Rue and Held (2005) under which we use the true model for a boundary set of thickness $m=2$ and the Gaussian Markov random-field approximation for the conditional distribution of the interior set. The drawback is the computational cost of calculating the boundary model.

Fig. 18 shows estimates of $\sigma^2\kappa^{2\nu}$, the consistently estimable parameter under the fixed domain asymptotics (Zhang, 2004). We observe a clear bias in the embedding estimates, whereas the naive approximation estimates have less bias but have high variability. We have not implemented the authors' approach using the Neumann boundary conditions.

Bo Li (*Purdue University, West Lafayette*) **and Marc G. Genton** (*Texas A&M University, College Station*)
In this very stimulating paper, the authors created a new path to deal with the computational challenge caused by large spatial data sets. Whereas most of the previous approaches mainly focused on screening out information that is relatively less important to gain computational efficiency (see Sun *et al.* (2011) for a recent review), this newly proposed method sought an explicit link between some Gaussian fields (GFs) and Gaussian Markov random fields (GMRFs) and thus enabled a direct application of the inherent

computational advantage in GMRFs to GFs. The GFs with Matérn covariance structure play a central role in spatial data modelling. Although the GMRF representation is developed only for the GFs with certain values of smoothness, we expect a wide application of this new approach since the smoothness parameter in the Matérn function is nevertheless difficult to estimate precisely. We genuinely appreciate the novelty and practical value of this paper. However, recently emerged data sets are often indexed by locations in both space and time, and many have more than one variable observed. Analyses with those data sets are more challenging owing to the cubic growth of computations in terms of the sample size. North *et al.* (2011) derived Matérn-like space–time correlations from evolving GFs governed by a white-noise-driven damped diffusion equation arising from simple energy balance climate models on the plane and on the sphere. It appears that those results could be used directly to extend the link between GFs and GMRFs to spatiotemporal data. Further extensions to a multivariate context remain open.

The authors gave an example of modelling non-stationary global temperature GFs and then making inference on the temperature process via GMRFs in conjunction with the integrated nested Laplace approximation. This can be very useful for the palaeoclimate community because one popular approach for large-scale palaeoclimate reconstructions is through Bayesian hierarchical models (Li *et al.*, 2010) where it is crucial to identify an appropriate model for the random process of climate variables. Such a model needs to be sufficiently flexible while still keeping the inference computationally feasible. The explicit link developed in this paper combined with the integrated nested Laplace approximation seems a promising direction. Since the proxy data used for the reconstruction carry various types of noise, a nugget effect may need to be considered. Would the approach be directly applicable if nuggets are included in the covariance model? An ambitious goal in palaeoclimate studies is simultaneously to reconstruct the entire space–time process of the temperature and other climate variables. Therefore, it again requires computational efficiency for spatiotemporal and multivariate data. We look forward to seeing further developments on this topic and in the mean time congratulate the authors for their outstanding work!

Georg Lindgren (*Lund University*)

I would like to add to the impressive list of applications of the Gaussian Markov random field–stochastic partial differential equation (SPDE) link, namely its use in ocean wave modelling. Traditionally, stochastic wave models have been based on linear Fourier analysis, possibly including low order interactions between the Fourier components. Such models are seen as approximations to the basic hydrodynamical (deterministic) partial differential equations for water waves. These equations have, in themselves, little room for stochastic forces.

One of the common spectra used in ocean modelling is the Pierson–Moskowitz spectrum

$$S(\omega,\theta)=A_{\mathrm{PM}}\omega^{-5}\exp(-B_{\mathrm{PM}}/\omega^{-4})\cos^{2s}(\theta-\theta_0).$$

The SPDE approach, developed in the paper presented, offers a promising link between the hydrodynamic and Fourier view on random ocean waves. It has recently been shown by David Bolin and Finn Lindgren (see Bolin and Lindgren (2011b) for the general theory and Bolin (2009) and Lindgren *et al.* (2010) for the wave application) that the solutions to a *nested* SPDE,

$$(\kappa^2-\Delta)^{(s+2)/2}X(\mathbf{t})=(\mathbf{B}^{\mathrm{T}}\nabla)^s\,W(\mathbf{t}),\qquad \mathbf{t}\in\mathbb{R}^2,$$

has a spectral density

$$S(\omega,\theta)=C\frac{\omega^{4s+3}}{(g^2\kappa^2+\omega^4)^{\alpha}}\cos^{2s}(\theta-\theta_0),$$

with g equal to Earth's acceleration. The vector $\mathbf{B}=(b_1,b_2)^{\mathrm{T}}$ determines the main direction θ_0 of the directional spectrum. For large s, this is close to the Pierson–Moskowitz wave spectrum and, thus, the SPDE approach could turn out to be a flexible alternative to the Fourier approach.

K. V. Mardia (*University of Leeds*)

I found the paper very timely and stimulating. The problem of dealing with large spatial data has a long history and the authors have given a comprehensive way forward. Mardia (2007) has given a historical background to the maximum likelihood methods for spatial data and pointed out that it seems there are still two main communities—one mining practitioners and the other mainstream statisticians.