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Supplementary material for 'Multivariate max-stable spatial processes'

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SUMMARY

This document contains technical details for deriving the multivariate max-stable spatial distributions described in the paper and simulation results for the trivariate Hüsler–Reiss process.

1. MULTIVARIATE HÜSLER–REISS PROCESS

1·1. *Proof of Proposition 1*

The convergence of $Z_n(s)$ to $Z(s)$ in the sense of finite-dimensional distributions is derived by showing that for a finite sequence of spatial locations $\{s_k\}_{k\in K} \in \mathcal{S}$, as $n \to \infty$

$$\mathrm{pr}[\{M_{in}(s_k) \le z_{in}(s_k)\}_{(i,k)\in J}] = (1 - Q_n[\{z_{in}(s_k)\}_{(i,k)\in J}])^n \to \exp\left(-V[\{z_i(s_k)\}_{(i,k)\in J}]\right),$$

where

$$Q_n[\{z_{in}(s_k)\}_{(i,k)\in J}] = \mathrm{pr}\{Y_{in}(s_k) > z_{in}(s_k), \text{for some } (i,k) \in J\},$$

$z_{in}(s_k) = z_i(s_k)/b_n + b_n$ is a sequence of real-valued constants and the normalizing constants are

$$b_n = (2\log n)^{1/2} - \frac{\log\log n + \log(4\pi)}{(2\log n)^{1/2}}.$$

Here $\{Y_{in}(s_k)\}_{(i,k)\in J}$ is a zero-mean, N-dimensional Gaussian random vector with cross-correlation matrix $\Sigma(s_k; n) = \{\rho_{ij}(s_k - s_l; n)\}_{(i,k),(j,l)\in J}$, $k \in K$, and the function V is an exponent function (de Haan & Ferreira, 2006, Ch. 6). In order to derive the exponent function V, the relation $nQ_n[\{z_{in}(s_k)\}_{(i,k)\in J}] \sim V[\{z_i(s_k)\}_{(i,k)\in J}]$ as $n \to \infty$ and the conditional tail dependence function framework (Nikoloulopoulos et al., 2009) are exploited. With uniform margins, that is, $x_i(s_k) = \log u_i(s_k)$ with $u_i(s_k) \in [0,1]$, $(i,k) \in J$, the function V is differentiable and is homogeneous of order 1. Thus, applying Euler's homogeneous theorem (Apostol, 1967), the exponent function can ve written as $V[\{x_i(s_k)\}_{(i,k)\in J}] = \sum_{(i,k)\in J} x_{ik}(\partial V/\partial x_{ik})$ for all $(x_{11}, \ldots, x_{pq}) \in \mathbb{R}_+^N$ with $x_{ik} \equiv x_i(s_k)$ (Nikoloulopoulos et al., 2009). The second-order partial derivatives of the distribution of $\{Y_{in}(s_k)\}_{(i,k)\in J}$ are continuous, then interchanging the order between the limit and the differentiation, after transforming the margins back to Gaussian margins, we obtain as $n \to \infty$

$$V[\{z_i(s_k)\}_{(i,k)\in J}] \sim \sum_{(i,k)\in J} e^{-z_i(s_k)} \mathrm{pr}[\{Y_{jn}(s_l) \leq z_{jn}(s_l)\}_{(j,l)\in J_{i,k}} | Y_{in}(s_k) = z_{in}(s_k)]. \quad (1)$$

Since

$$\{Y_{jn}(s_l) \leq z_{jn}(s_l)\}_{(j,l)\in J_{i,k}} \left| Y_{in}(s_k) = z_{in}(s_k) \sim \mathcal{N}_{N-1}\left(\Sigma_{ij}(s_l;n) z_{in}(s_k), \tilde{\Sigma}_{j|i}(s_l|s_k;n)\right)\right.,$$

where $\tilde{\Sigma}_{j|i}(s_l|s_k;n) = \Sigma_{jj}(s_l;n) - \Sigma_{ji}(s_l;n)\Sigma_{ii}^{-1}(s_k;n)\Sigma_{ij}(s_l;n)$ is a $(N-1)\times(N-1)$ partial correlation matrix with the generic entry $\rho_{jv}(s_l - s_w;n) - \rho_{ji}(s_l - s_k;n)\rho_{vi}(s_w - s_k;n)$ with $(i,k) \in J$ and $(j,l),(v,w) \in J_{i,k}$. Here, $\Sigma_{jj}(s_l;n)$ is the correlation matrix of $\{Y_{jn}(s_l)\}_{(j,l)\in J_{i,k}}$, $\Sigma_{ii}(s_k;n)$ is the correlation of $Y_{in}(s_k)$ and $\Sigma_{ji}(s_l;n)$ is the matrix of pairwise correlations between $Y_{in}(s_k)$ and each element of the sequence $\{Y_{jn}(s_l)\}_{(j,l)\in J_{i,k}}$. Therefore,

$$\begin{aligned} &\mathrm{pr}[\{Y_{jn}(s_l) \leq z_{jn}(s_l)\}_{(j,l)\in J_{i,k}} | Y_{in}(s_k) = z_{in}(s_k)] \\ &= \Phi_{N-1,\bar{\Sigma}_{i|j}(s_l|s_k;n)}\left[\left\{\frac{z_{jn}(s_l) - \rho_{ij}(s_k - s_l;n) z_{in}(s_k)}{\{1-\rho_{ij}^2(s_k - s_l;n)\}^{1/2}}\right\}_{(j,l)\in J_{i,k}}\right], \end{aligned}$$

where $\Phi_{N-1,\bar{\Sigma}_{i|j}(s_l|s_k;n)}$ is an $(N-1)$-dimensional Gaussian distribution with zero-mean and partial correlation matrix $\bar{\Sigma}_{i|j}(s_l|s_k;n)$, where for the generic entry, it holds that

$$\frac{\rho_{jv}(s_l - s_w;n) - \rho_{ji}(s_l - s_k;n)\rho_{vi}(s_w - s_k;n)}{[\{1-\rho_{ji}^2(s_l - s_r;n)\}\{1-\rho_{vi}^2(s_w - s_k;n)\}]^{1/2}} \to \frac{\lambda_{ji}^2(s_l - s_k) + \lambda_{vi}^2(s_w - s_k) - \lambda_{jv}^2(s_l - s_w)}{2\lambda_{ji}(s_l - s_r)\lambda_{vi}(s_w - s_k)},$$

as $n \to \infty$. Then, as $n \to \infty$, formula (1) becomes

$$\sum_{(i,k)\in J} e^{-z_i(s_k)} \Phi_{N-1,\bar{\Lambda}_{ik}}\left[\left\{\frac{\lambda_{ij}(s_k - s_l)}{2} + \frac{z_j(s_l) - z_i(s_k)}{\lambda_{ij}(s_k - s_l)}\right\}_{(j,l)\in J_{i,k}}\right], \quad (2)$$

where the partial correlation matrix $\bar{\Lambda}_{ik}$ is equal to

$$\begin{pmatrix} 1 & \cdots & \frac{\lambda^2_{i;k+1,k}+\lambda^2_{i;k,q}-\lambda^2_{i;k+1,q}}{2\lambda_{i;k+1,k}\lambda_{i;q+k}} & \frac{\lambda^2_{i;k+1,k}+\lambda^2_{i,i+1}-\lambda^2_{i,i+1;k+1,k}}{2\lambda_{i;k+1,k}\lambda_{i+1,i}} & \cdots & \frac{\lambda^2_{i;k+1,k}+\lambda^2_{i,p;k,q}-\lambda^2_{i,p;k+1,q}}{2\lambda_{i;k+1,k}\lambda_{p,i;q,k}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda^2_{i;q,k}+\lambda^2_{i;k,k+1}-\lambda^2_{i;q,k+1}}{2\lambda_{i;k+1,k}\lambda_{i;q,k}} & \cdots & 1 & \frac{\lambda^2_{i;q,k}+\lambda^2_{i,i+1}-\lambda^2_{i,i+1;q,k}}{2\lambda_{i;q,k}\lambda_{i,i+1}} & \cdots & \frac{\lambda^2_{i;q,k}+\lambda^2_{i,p;k,q}-\lambda^2_{i,p}}{2\lambda_{i;q,k}\lambda_{p,i;q,k}} \\ \frac{\lambda^2_{i+1,i}+\lambda^2_{i;k,k+1}-\lambda^2_{i+1,i;k,k+1}}{2\lambda_{i;k+1,k}\lambda_{i+1,i}} & \cdots & \frac{\lambda^2_{i+1,i}+\lambda^2_{i;k,q}-\lambda^2_{i+1,i;k,q}}{2\lambda_{i;q,k}\lambda_{i+1,i}} & 1 & \cdots & \frac{\lambda^2_{i+1,i}+\lambda^2_{i,p;k,q}-\lambda^2_{i+1,p;k,q}}{2\lambda_{i+1,i}\lambda_{p,i;q,k}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda^2_{p,i;q,k}+\lambda^2_{i;k,k+1}-\lambda^2_{p,i;q,k+1}}{2\lambda_{i;k+1,k}\lambda_{p,i;q,k}} & \cdots & \frac{\lambda^2_{p,i;q,k}+\lambda^2_{i;k,q}-\lambda^2_{p,i}}{2\lambda_{i;q,k}\lambda_{p,i;q,k}} & \frac{\lambda^2_{p,i;q,k}+\lambda^2_{i,i+1}-\lambda^2_{p,i+1;q,k}}{2\lambda_{i+1,i}\lambda_{p,i;q,k}} & \cdots & 1 \end{pmatrix}, \quad (3)$$

with $\lambda_{i,j;k,l} \equiv \lambda_{ij}(s_k - s_l)$, $(i,k) \in J$. Finally, taking the transformation $z_i(s_k) = \log\{\tilde{z}_i(s_k)\}$, $(i,k) \in J$, we obtain the exponent function of Proposition 1. It is immediate to check that a distribution with that exponent function has unit Fréchet margins and is max-stable, and this completes the proof. $\square$

1·2. *Tri- and four-dimensional Hüsler–Reiss distributions*

Particular examples of Hüsler–Reiss distributions, which can be useful for practical purposes, are reported next. These can be easily deduced from the exponent function (2).

Example 1. Consider two variables (Z_i, Z_j), $i, j \in I$. Then for any spatial point $s \in \mathcal{S}$, separation $h \in \mathbb{R}$ and positive thresholds z_i, $i = 1, \dots, 4$, the Hüsler–Reiss distribution for the sequence $\{Z_i(s), Z_j(s), Z_i(s+h), Z_j(s+h)\}$ is

$$\begin{aligned}\exp\Bigg[& z_1^{-1}\Phi_{3,\bar{\Lambda}_1}\left\{\frac{\lambda_i(h)}{2}+\frac{\log z_2/z_1}{\lambda_i(h)}, \frac{\lambda_{ij}}{2}+\frac{\log z_3/z_1}{\lambda_{ij}}, \frac{\lambda_{ij}(h)}{2}+\frac{\log z_4/z_1}{\lambda_{ij}(h)}\right\} \\ & + z_2^{-1}\Phi_{3,\bar{\Lambda}_2}\left\{\frac{\lambda_i(h)}{2}+\frac{\log z_1/z_2}{\lambda_i(h)}, \frac{\lambda_{ji}(h)}{2}+\frac{\log z_3/z_2}{\lambda_{ji}(h)}, \frac{\lambda_{ij}}{2}+\frac{\log z_4/z_2}{\lambda_{ij}}\right\} \\ & + z_3^{-1}\Phi_{3,\bar{\Lambda}_3}\left\{\frac{\lambda_j(h)}{2}+\frac{\log z_4/z_3}{\lambda_j(h)}, \frac{\lambda_{ij}}{2}+\frac{\log z_1/z_3}{\lambda_{ij}}, \frac{\lambda_{ji}(h)}{2}+\frac{\log z_2/z_3}{\lambda_{ji}(h)}\right\} \\ & + z_4^{-1}\Phi_{3,\bar{\Lambda}_4}\left\{\frac{\lambda_j(h)}{2}+\frac{\log z_3/z_4}{\lambda_j(h)}, \frac{\lambda_{ij}(h)}{2}+\frac{\log z_1/z_4}{\lambda_{ij}(h)}, \frac{\lambda_{ij}}{2}+\frac{\log z_2/z_4}{\lambda_{ij}}\right\}\Bigg],\end{aligned}$$

where in the expression we used the equivalences $\lambda_{ji} = \lambda_{ij}$, $\lambda_i(-h) = \lambda_i(h)$, $\lambda_j(-h) = \lambda_j(h)$, $\lambda_{ij}(-h) = \lambda_{ji}(h)$ and $\lambda_{ji}(-h) = \lambda_{ij}(h)$. Note that $\lambda_{ij}(h) \neq \lambda_{ji}(h)$. Specifically, the expression of the 3×3 symmetric partial correlation matrix-valued functions are

$$\bar{\Lambda}_1 = \begin{pmatrix} 1 & \frac{\lambda_{ij}^2+\lambda_i^2(h)-\lambda_{ji}^2(h)}{2\lambda_{ij}\lambda_i(h)} & \frac{\lambda_i^2(h)+\lambda_{ij}^2(h)-\lambda_{ij}^2}{2\lambda_i(h)\lambda_{ij}(h)} \\ & 1 & \frac{\lambda_{ij}^2+\lambda_{ij}^2(h)-\lambda_j^2(h)}{2\lambda_{ij}\lambda_{ij}(h)} \\ & & 1 \end{pmatrix}, \quad \bar{\Lambda}_2 = \begin{pmatrix} 1 & \frac{\lambda_i^2(h)+\lambda_{ji}^2(h)-\lambda_{ij}^2}{2\lambda_i(h)\lambda_{ji}(h)} & \frac{\lambda_{ij}^2+\lambda_i^2(h)-\lambda_{ij}^2(h)}{2\lambda_{ij}\lambda_i(h)} \\ & 1 & \frac{\lambda_{ij}^2+\lambda_{ji}^2(h)-\lambda_j^2(h)}{2\lambda_{ij}\lambda_{ji}(h)} \\ & & 1 \end{pmatrix},$$

$$\bar{\Lambda}_3 = \begin{pmatrix} 1 & \frac{\lambda_{ij}^2+\lambda_{ji}^2(h)-\lambda_i^2(h)}{2\lambda_{ij}\lambda_{ji}(h)} & \frac{\lambda_{ij}^2+\lambda_j^2(h)-\lambda_{ij}^2(h)}{2\lambda_{ij}\lambda_j(h)} \\ & 1 & \frac{\lambda_j^2+\lambda_{ji}^2(h)-\lambda_{ij}}{2\lambda_j(h)\lambda_{ji}(h)} \\ & & 1 \end{pmatrix}, \quad \bar{\Lambda}_4 = \begin{pmatrix} 1 & \frac{\lambda_{ij}^2+\lambda_{ij}^2(h)-\lambda_i^2(h)}{2\lambda_{ij}\lambda_{ij}(h)} & \frac{\lambda_j^2+\lambda_{ij}^2(h)-\lambda_{ij}}{2\lambda_j(h)\lambda_{ij}(h)} \\ & 1 & \frac{\lambda_{ij}^2+\lambda_j^2(h)-\lambda_{ji}^2(h)}{2\lambda_{ij}\lambda_j(h)} \\ & & 1 \end{pmatrix}.$$

Example 2. Consider three variables (Z_i, Z_j, Z_v), $i, j, v \in I$. Then for any spatial point $s \in \mathcal{S}$, separations $h, h' \in \mathbb{R}$ and positive thresholds z_i, $i = 1, \dots, 3$, the Hüsler–Reiss distribution for the sequence $\{Z_i(s), Z_j(s+h), Z_v(s+h')\}$ is

$$\begin{aligned}\exp\Bigg[& \frac{1}{z_1}\Phi_{2,\bar{\Lambda}_1}\left\{\frac{\lambda_{ij}(h)}{2}+\frac{\log\frac{z_2}{z_1}}{\lambda_{ij}(h)}, \frac{\lambda_{iv}(h')}{2}+\frac{\log\frac{z_3}{z_2}}{\lambda_{iv}(h')}\right\} + \frac{1}{z_2}\Phi_{2,\bar{\Lambda}_2}\left\{\frac{\lambda_{ij}(h)}{2}+\frac{\log\frac{z_2}{z_2}}{\lambda_{ij}(h)}, \frac{\lambda_{jv}(h'')}{2}+\frac{\log\frac{z_3}{z_2}}{\lambda_{iv}(h'')}\right\} \\ & + \frac{1}{z_3}\Phi_{2,\bar{\Lambda}_3}\left\{\frac{\lambda_{iv}(h')}{2}+\frac{\log\frac{z_1}{z_2}}{\lambda_{iv}(h')}, \frac{\lambda_{jv}(h'')}{2}+\frac{\log\frac{z_2}{z_3}}{\lambda_{jv}(h'')}\right\}\Bigg],\end{aligned}$$

where $h'' \in \mathbb{R}$ and in the expression we used the equivalences $\lambda_{ji}(-h) = \lambda_{ij}(h)$, $\lambda_{vi}(-h') = \lambda_{iv}(h')$ and $\lambda_{vj}(-h'') = \lambda_{jv}(h'')$. Specifically, the expression of the symmetric 2×2 partial correlation matrix-valued functions are

$$\bar{\Lambda}_1 = \begin{pmatrix} 1 & \frac{\lambda_{ij}^2(h)+\lambda_{iv}^2(h')-\lambda_{jv}^2(h'')}{2\lambda_{ij}(h)\lambda_{iv}(h')} \\ & 1 \end{pmatrix}, \bar{\Lambda}_2 = \begin{pmatrix} 1 & \frac{\lambda_{ij}^2(h)+\lambda_{jv}^2(h'')-\lambda_{iv}^2(h')}{2\lambda_{ij}(h)\lambda_{jv}(h'')} \\ & 1 \end{pmatrix}, \bar{\Lambda}_3 = \begin{pmatrix} 1 & \frac{\lambda_{jv}^2(h'')+\lambda_{iv}^2(h')-\lambda_{ij}^2(h)}{2\lambda_{iv}(h')\lambda_{jv}(h'')} \\ & 1 \end{pmatrix}.$$

2. FINITE DIMENSIONAL DISTRIBUTION OF THE MULTIVARIATE EXTREMAL-t PROCESS

Similar to the Gaussian case, it is required to show that as $n \to \infty$,

$$\mathrm{pr}[\{M_{in}(s_k) \leq z_{in}(s_k)\}_{(i,k)\in J}] = (1 - Q_n[\{z_{in}(s_k)\}_{(i,k)\in J}])^n \to \exp\left(-V[\{z_i(s_k)\}_{(i,k)\in J}]\right),$$

where

$$Q_n[\{z_{in}(s_k)\}_{(i,k)\in J}] = \text{pr}\{Y_i(s_k) > z_{ni}(s_k), \text{for some } (i,k) \in J\},$$

where $z_{in}(s_k) = a_n\{z_i(s_k)\}^{1/\nu}$. The normalizing constants a_n are obtained from the equation $a_n = T_\nu^{-1}(1-1/n)$, where T_ν^{-1} is the inverse of the standard univariate Student-t distribution with $\nu > 0$ degrees of freedom. These, for large n, are

$$a_n = \left[\frac{n\,\nu^{(\nu-2)/2}\Gamma\{(\nu+1)/2\}}{\Gamma(\nu/2)\sqrt{\pi}}\right]^{1/\nu}.$$

The sequence $\{Y_i(s_k)\}_{(i,k)\in J}$ has an N-dimensional Student-t distribution with zero centers, dispersion matrix $\Sigma = \{\rho_{ij}(s_k - s_l)\}_{(i,k),(j,l)\in J}$ and $\nu > 0$ degrees of freedom. Applying the conditional tail dependence function framework (Nikoloulopoulos et al., 2009), as $n \to \infty$ it follows that

$$V[\{z_i(s_k)\}_{(i,k)\in J}] \sim \sum_{(i,k)\in J} \frac{1}{z_i(s_k)} \text{pr}[\{Y_j(s_l) \le z_{jn}(s_l)\}_{(j,l)\in J_{i,k}} | Y_i(s_k) = z_{in}(s_k)],$$

where

$$\{Y_j(s_l) \le z_{jn}(s_l)\}_{(j,l)\in J_{i,k}} \left| Y_i(s_k) = z_{in}(s_k) \sim \mathcal{T}_{N-1}\left(\Sigma_{ij}(s_l) z_{in}(s_k), \tilde{\Sigma}_{j|i}(s_l|s_k), \nu+1\right)\right.,$$

with $\mathcal{T}_{N-1}$ denoting an $(N-1)$-dimensional Student-t distribution with $\nu+1$ degrees of freedom and $\Sigma_{ij}(s_l)$ and $\tilde{\Sigma}_{j|i}(s_l|s_k)$ are similar to those appearing in (1) but do not depend on n. We have

$$\begin{aligned}
&\text{pr}[\{Y_j(s_l) \le z_{jn}(s_l)\}_{(j,l)\in J_{i,k}} | Y_i(s_k) = z_{in}(s_k)] \\
&= T_{N-1,\bar{\Sigma}_{ik},\nu+1}\left[\left\{\left(\frac{\nu+1}{[\nu + \{z_{in}(s_k)\}^2]\{1-\rho_{ij}^2(s_k - s_l)\}}\right)^{1/2} \{z_{jn}(s_l) - z_{in}(s_k)\rho_{ij}(s_k - s_l)\}\right\}_{(j,l)\in J_{i,k}}\right],
\end{aligned}$$

where $T_{N-1,\bar{\Sigma}_{ik},\nu+1}$ is an $(N-1)$-dimensional Student-t distribution with $\nu+1$ degrees of freedom, zero centers and partial correlation matrix $\bar{\Sigma}_{ik}$. The latter is equal to

$$\begin{pmatrix}
1 & \cdots & \frac{\rho_{i;k+1,q} - \rho_{i;k+1,k}\rho_{i;k,q}}{\sqrt{1-\rho^2_{i;k+1,k}}\sqrt{1-\rho^2_{i;q+k}}} & \frac{\rho_{i,i+1;k+1,k} - \rho_{i;k+1,k}\rho_{i,i+1}}{\sqrt{1-\rho^2_{i;k+1,k}}\sqrt{1-\rho^2_{i+1,i}}} & \cdots & \frac{\rho_{i,p;k+1,q} - \rho_{i;k+1,k}\rho_{i,p;k,q}}{\sqrt{1-\rho^2_{i;k+1,k}}\sqrt{1-\rho^2_{p,i;q,k}}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\rho_{i;q,k+1} - \rho_{i;q,k}\rho_{i;k,k+1}}{\sqrt{1-\rho^2_{i;k+1,k}}\sqrt{1-\rho^2_{i;q,k}}} & \cdots & 1 & \frac{\rho_{i,i+1;q,k} - \rho_{i;q,k}\rho_{i,i+1}}{\sqrt{1-\rho^2_{i;q,k}}\sqrt{1-\rho^2_{i,i+1}}} & \cdots & \frac{\rho_{i,p} - \rho_{i;q,k}\rho_{i,p;k,q}}{\sqrt{1-\rho^2_{i;q,k}}\sqrt{1-\rho^2_{p,i;q,k}}} \\
\frac{\rho_{i+1,i;k,k+1} - \rho_{i+1,i}\rho_{i;k,k+1}}{\sqrt{1-\rho^2_{i;k+1,k}}\sqrt{1-\rho^2_{i+1,i}}} & \cdots & \frac{\rho_{i+1,i;k,q} - \rho_{i+1,i}\rho_{i;k,q}}{\sqrt{1-\rho^2_{i;q,k}}\sqrt{1-\rho^2_{i+1,i}}} & 1 & \cdots & \frac{\rho_{i+1,p;k,q} - \rho_{i+1,i}\rho_{i,p;k,q}}{\sqrt{1-\rho^2_{i+1,i}}\sqrt{1-\rho^2_{p,i;q,k}}} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\rho_{p,i;q,k+1} - \rho_{p,i;q,k}\rho_{i;k,k+1}}{\sqrt{1-\rho^2_{i;k+1,k}}\sqrt{1-\rho^2_{p,i;q,k}}} & \cdots & \frac{\rho_{p,i} - \rho_{p,i;q,k}\rho_{i;k,q}}{\sqrt{1-\rho^2_{i;q,k}}\sqrt{1-\rho^2_{p,i;q,k}}} & \frac{\rho_{p,i+1;q,k} - \rho_{p,i;q,k}\rho_{i,i+1}}{\sqrt{1-\rho^2_{i+1,i}}\sqrt{1-\rho^2_{p,i;q,k}}} & \cdots & 1
\end{pmatrix},$$

where $\rho_{i,j;k,l} \equiv \rho_{ij}(s_k - s_l)$, $(i,k),(j,l) \in J$. Then for $n \to \infty$ the exponent function $V[\{z_i(s_k)\}_{(i,k)\in J}]$ is equal to

$$\sum_{(i,k)\in J} \frac{1}{z_i(s_k)}\, T_{N-1,\bar{\Sigma}_{ik},\nu+1}\left\{\left(\left\{\frac{\nu+1}{1-\rho_{ij}^2(s_k - s_l)}\right\}^{1/2}\left[\{z_j(s_l)/z_i(s_k)\}^{1/\nu} - \rho_{ij}(s_k - s_l)\right]\right)_{(j,l)\in J_{ik}}\right\}.$$

3. FINITE DIMENSIONAL DISTRIBUTION OF THE MULTIVARIATE BROWN–RESNICK PROCESS

For a finite sequence of spatial locations $\{s_k\}_{k\in K} \in \mathcal{S}$ and $z_i(s_k) > 0$ for all $(i,k) \in J$, the distribution of the multivariate Brown–Resnick process is derived computing the exponent function (see equation (9) in §3.1 of the paper)

$$V[\{z_i(s_k)\}_{(i,k)\in J}] = E\left[\max_{(i,k)\in J}\{W_i(s_k)/z_i(s_k)\}\right], \tag{4}$$

where $W(s) = \exp\{X(s) - \sigma^2(s)/2\}$, $s \in \mathcal{S}$ and $X(s)$ and $\sigma^2(s)$ are defined as in §3.2 of the paper. The expectation (4) can be written as

$$\int_0^\infty \cdots \int_0^\infty \max_{(i,k)\in J}\{w_{ik}/z_i(s_k)\}\, f(w_{11},\ldots,w_{pq})\mathrm{d}w_{11}\cdots\mathrm{d}w_{pq} = \sum_{(i,k)\in J} Q_{ik}/z_i(s_k)$$

where f is the density function of the p-dimensional process W observed at q locations and $w_{ik} \equiv w_i(s_k)$. For each fixed $(i,k) \in J$, setting $w = \{w_{jl}, (j,l) \in J_{ik}\}$ and $z(s) = \{z_j(s_l), (j,l) \in J_{ik}\}$, the quantity Q_{ik} is

$$Q_{ik} = \int_0^\infty \int_0^{wz(s)/z_i(s_k)} f(w)\mathrm{d}w\,\mathrm{d}w_{ik}. \tag{5}$$

As shown by Huser & Davison (2013), the solution of (5) is

$$\begin{aligned} Q_{ik} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{x_{ik}-c} \exp(x_{ik} - \sigma_i^2(s_k)/2)\phi_{N,\Sigma}(x)\mathrm{d}x\,\mathrm{d}x_{ik} \\ &= \cdots = \Phi_{N-1,\bar{\Sigma}_{ik}}\left(\left[\left\{\frac{\gamma_{ij}(s_k-s_l)}{2}\right\}^{1/2} + \frac{\log\{z_j(s_l)/z_i(s_k)\}}{\{2\gamma_{ij}(s_k-s_l)\}^{1/2}}\right]_{(j,l)\in J_{ik}}\right), \end{aligned}$$

where $c = \sigma_i^2(s_k)/2 - \sigma^2(s)/2 + \log\{z_{ik}/z(s)\}$ with $\sigma^2(s) = \{\sigma_l^2(s_l), (j,l) \in J_{ik}\}$ and with all the ratios taken componentwise, $\phi_{N,\Sigma}$ is an N-dimensional Gaussian density centered at zero with an appropriate covariance matrix Σ. Here $\Phi_{N-1,\bar{\Sigma}_{ik}}$ is an $(N-1)$-dimensional Gaussian distribution with zero-mean and partial correlation matrix $\bar{\Sigma}_{ik}$. Specifically, the expression of the latter is (3), with $\lambda_{ij}(s_k - s_l) = \{2\gamma_{ij}(s_k - s_l)\}^{1/2}$. In the inner integral, if the order of the variables is changed, then the solution remains the same due to the symmetry of the density function of $X(s)$. In conclusion, the exponent function of the multivariate Brown–Resnick process is

$$\sum_{(i,k)\in J} \frac{1}{z_i(s_k)}\Phi_{N-1,\bar{\Sigma}_{ik}}\left(\left[\left\{\frac{\gamma_{ij}(s_k-s_l)}{2}\right\}^{1/2} + \frac{\log\{z_j(s_l)/z_i(s_k)\}}{\{2\gamma_{ij}(s_k-s_l)\}^{1/2}}\right]_{(j,l)\in J_{ik}}\right).$$

Remark 1. Similarly to the univariate case, the multivariate Hüsler–Reiss model (see §2.1) emerges as special cases of the extremal-*t* model (see §2.3), assuming that the pairwise correlations, $\rho_{ij}(s_k - s_l; \nu) = 1 - \lambda_{ij}^2(s_k - s_l)/(4\nu)$, for all $i, j \in I$ and $k, l \in K$, tend to 1 as $\nu \to \infty$ (Nikoloulopoulos et al., 2009). The multivariate extremal-Gaussian model emerges as a special case of the extremal-*t* for $\nu = 1$ (Opitz, 2013). The multivariate extremal-Gaussian model is obtained defining $W(s) = \max\{0, X(s)\}$ in equation (9) of §3.1, where $X(s)$ is a zero-mean, unit-variance, p-dimensional multivariate Gaussian process with a matrix-valued covariance function, $\Sigma(h) = \{\rho_{ij}(h)\}_{i,j\in I}$, and with the maximum taken componentwise; see Schlather (2002) for the univariate case.

4. BIVARIATE DISTRIBUTION OF THE MULTIVARIATE SMITH GAUSSIAN EXTREME-VALUE PROCESS

A definition of a multivariate Gaussian extreme-value process is obtained defining, in equation (9) of §3.1, $W(s) = \{f_1(X^{(m)} - s), \ldots, f_p(X^{(m)} - s)\}$, where f_i, $i \in I$, is a unimodal continuous probability density on $\mathbb{R}^d$, $\{X^{(m)}\}_{m\geq 1}$ are points of a homogeneous Poisson process on $\mathbb{R}^d$, with intensity measure $\delta(dx)$, and $\delta(dx)$ is a positive measure.

PROPOSITION 1. *Let $d = 2$, δ be the Lebesgue measure and f be the uncorrelated bivariate normal density*

$$\phi_i(x/\sigma_i) = (2\pi)^{-1/2}\sigma_i^{-1} \exp\left\{-\|x\|^2/(2\sigma_i^2)\right\},\ x = (x_1, x_2) \in \mathbb{R}^2,\ i = 1, 2. \tag{6}$$

For any $s \in \mathcal{S}$, $h = (h_1, h_2) \in \mathbb{R}^2$ and $z_1, z_2 > 0$ then,

$$pr(Z_1(s) \leq z_1, Z_2(s + h) \leq z_2) = \exp\left\{-V_{12}^{[2]}(z_1, z_2)\right\},$$

where

$$V_{12}^{[2]}(z_1, z_2) = \begin{cases} 1/z_2, & 0 < z_2 < c(u, \sigma_1; \|h\|)\, z_1, \quad u > 1, \\ \frac{pr(X_1 \in A_1)}{z_1} + \frac{pr(X_2 \in A_2^c)}{z_2}, & z_1\, c(u, \sigma_1; \|h\|) \leq z_2 \leq z_1, \quad u > 1, \\ \frac{pr(X_2 \in A_2)}{z_2} + \frac{pr(X_1 \in A_1^c)}{z_1}, & z_1 \leq z_2 < c(u, \sigma_1; \|h\|)\, z_1, \quad u < 1, \\ 1/z_1, & z_2 \geq c(u, \sigma_1; \|h\|)\, z_1, \quad u < 1, \end{cases}$$

$z_1(s) \equiv z_1$ and $z_2(s + h) \equiv z_2$, X_1 and X_2 are random vectors with bivariate density (6)*, $u = \sigma_2^2/\sigma_1^2$, $c(u, \sigma_1; \|h\|) = u^{-1} \exp[\|h\|^2/\{2\sigma_1^2(1 - u)\}]$,*

$$A_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 : \left(x_1 - \frac{h_1}{1 - u}\right)^2 + \left(x_2 - \frac{h_2}{1 - u}\right)^2 \leq \frac{u\|h\|^2}{(1 - u)^2} + \frac{2\sigma_2^2}{1 - u} \log\left(\frac{z_1}{uz_2}\right)\right\},$$

and

$$A_2 = \left\{(x_1, x_2) \in \mathbb{R}^2 : \left(x_1 - \frac{uh_1}{1 - u}\right)^2 + \left(x_2 - \frac{uh_2}{1 - u}\right)^2 \leq \frac{u\|h\|^2}{(1 - u)^2} + \frac{2\sigma_2^2}{1 - u} \log\left(\frac{z_1}{uz_2}\right)\right\}.$$

Proof. Start with the case $d = 1$. For any $h \in \mathbb{R}$ and $z_1, z_2 > 0$ then,

$$V_{12}^{[2]}(z_1, z_2) = \int_{-\infty}^{\infty} \max\left[\phi(x/\sigma_1)/z_1, \phi\{(x - h)/\sigma_2\}/z_2\right] \mathrm{d}x = Q_1/z_1 + Q_2/z_2,$$

where, with $I(\cdot)$ denoting the indicator function,

$$Q_1 = \int_{-\infty}^{\infty} \phi\,(x/\sigma_1)\, I\left[\phi(x/\sigma_1)/z_1 \geq \phi\{(x - h)/\sigma_2\}/z_2\right] \mathrm{d}x,$$

and Q_2 is similar to Q_1 but with $\phi\{(x - h)/\sigma_1\}$ instead of $\phi(x/\sigma_1)$ and the inverted relation in the indicator function. With similar arguments as de Haan & Pereira (2006), Q_1 depends on

$$\phi(x/\sigma_1)/z_i \geq \phi\{(x - h)/\sigma_2\}/z_2 \Longleftrightarrow (1 - u)x^2 - 2\,h\,u + d \geq 0,$$

where $u = \sigma_2^2/\sigma_1^2$ and $d = h^2 - 2\sigma_2^2 \log\{z_1/(uz_2)\}$. Then, Q_1 assumes three different values, depending on when the inequality is satisfied. Assume the condition $1 - u > 0$, which is satisfied if and only if $0 < \sigma_2^2 < \sigma_1^2$. In addition, assume the condition $h^2 - (1 - u)d \leq 0$, which is satisfied if and only if $z_2 \geq c(u, \sigma_1; h)z_1$, where $c(u, \sigma_1; h) = u^{-1} \exp[h^2/\{2\sigma_1^2(1 - u)\}]$.

Then, $Q_1 = 1$. In the case that the opposite inequality, $z_2 < c(u, \sigma_1; h)z_1$, is satisfied, then

$$(1-u)x^2 - 2\,h\,u + d \geq 0 \Longleftrightarrow \left(x - \frac{h}{1-u}\right)^2 \geq k,$$

where

$$k = \frac{u\,h^2}{(1-u)^2} + \frac{2\sigma_2^2}{1-u}\log\left(\frac{z_1}{u^{1/2}z_2}\right).$$

The right-hand side inequality is equivalent to

$$\left|x - \frac{h}{1-u}\right| \geq k$$

and this is verified by the values x such that $\{x < L\} \cup \{x > U\}$, where

$$L = h/(1-u) - k, \quad U = h/(1-u) + k.$$

Thus, $Q_1 = \Phi(L/\sigma_1) + 1 - \Phi(U/\sigma_1)$. On the other hand, if we assume that $1 - u < 0$, which implies that $0 < \sigma_1^2 < \sigma_2^2$, then $(1-u)x^2 - 2\,h\,u + d \geq 0 \Longleftrightarrow L \leq x \leq U$, where $k > 0 \Longleftrightarrow z_1\,c(u, \sigma_1; h) < z_2$. Therefore, $Q_1 = \Phi(U/\sigma_1) - \Phi(L/\sigma_1)$. Solving the integral Q_2 in a similar way, we obtain, if $0 < \sigma_2^2 < \sigma_1^2$ and $z_1 \leq z_2 < c(u, \sigma_1; h)z_1$ then, $Q_2 = \Phi(U'/\sigma_2) - \Phi(L'/\sigma_2)$, where

$$L' = hu/(1-u) - k, \quad U' = hu/(1-u) + k.$$

If $0 < \sigma_1^2 < \sigma_2^2$ and $z_1 c(u, \sigma_1; h) < z_2 \leq z_1$, then $Q_2 = \Phi(L'/\sigma_2) + 1 - \Phi(U'/\sigma_2)$. Finally, if $0 < z_2 < c(u, \sigma_1; h)z_1$, then $Q_2 = 1$. Combining the solutions together, we obtain

$$V_{12}^{[2]}(z_1, z_2) = \begin{cases} 1/z_2, & 0 < z_2 < c(u, \sigma_1; h)\,z_1, \quad u > 1, \\ \dfrac{\Phi\left(\frac{U}{\sigma_1}\right) - \Phi\left(\frac{L}{\sigma_1}\right)}{z_1} + \dfrac{\Phi\left(\frac{L'}{\sigma_2}\right) + \Phi\left(-\frac{U'}{\sigma_2}\right)}{z_2}, & z_1\,c(u, \sigma_1; h) \leq z_2 \leq z_1, \quad u > 1, \\ \dfrac{\Phi\left(\frac{U'}{\sigma_2}\right) - \Phi\left(\frac{L'}{\sigma_2}\right)}{z_2} + \dfrac{\Phi\left(\frac{L}{\sigma_1}\right) + \Phi\left(-\frac{U}{\sigma_1}\right)}{z_1}, & z_1 \leq z_2 < c(u, \sigma_1; h)\,z_1, \quad u < 1, \\ 1/z_1, & z_2 \geq c(u, \sigma_1; h)\,z_1, \quad u < 1. \end{cases}$$

If $d = 2$, the proof is analogous to the unidimensional case. □

5. Simulation results for trivariate Hüsler–Reiss process

5·1. *Bivariate case*

Table 1 shows the estimation results concerning the first simulation study in §5, that have not been reported for brevity. The study is based on 1000 simulations. Each estimate is obtained with 30 independent replicates of a bivariate Hüsler–Reiss random field generated over 35 uniformly distributed points on $[0, 100]^2$.

From the table we can see that, for all the parameter configurations that we considered, the points estimates, obtained with the estimation methods: multivariate extremal-coefficient, multivariate F-madogram, pairwise likelihood (10), triplewise likelihood (11) with D_3^* and weighted triplewise likelihood (12), are centered at the true parameter values. According to the bootstrap standard errors, we see that the new weighted composite likelihood estimator provides accurate estimates with a relative small sample size.

Table 1. *Estimation results of bivariate Hüsler–Reiss process simulations. The row "True" reports the true parameter values. The other rows reports the estimates based on the extremal-coefficient (θ_{CI}), the F-madogram ($\nu_{F\text{-}CI}$), the pairwise likelihood ($\ell_{2\text{-}CI}$) using all pairs, the pairwise likelihood using cross-variable-different-location ($\ell_{2\text{-}C}$), and the new weighted composite likelihood ($\ell_{3\text{-}W}$). The value in parenthesis below each point estimate is the bootstrap standard error.*

	α	κ	λ_{12}	α	κ	λ_{12}	α	κ	λ_{12}
True	30	0.5	0.3	30	0.3	0.8	5	0.3	0.3
θ_{CI}	33.75	0.51	0.30	44.78	0.31	0.79	7.07	0.33	0.33
	(20.48)	(0.07)	(0.05)	(37.06)	(0.09)	(0.18)	(6.77)	(0.12)	(0.06)
$\nu_{F\text{-CI}}$	32.84	0.52	0.30	39.21	0.32	0.81	6.78	0.32	0.29
	(20.02)	(0.09)	(0.05)	(32.97)	(0.06)	(0.17)	(4.54)	(0.07)	(0.04)
$\ell_{2\text{-C}}$	31.00	0.57	0.27	33.80	0.42	0.81	9.13	0.39	0.36
	(19.28)	(0.19)	(0.27)	(37.00)	(0.31)	(0.36)	(12.63)	(0.21)	(0.37)
$\ell_{2\text{-CI}}$	28.84	0.52	0.30	28.64	0.32	0.84	5.02	0.32	0.30
	(17.79)	(0.07)	(0.05)	(22.57)	(0.06)	(0.18)	(3.33)	(0.06)	(0.04)
$\ell_{3\text{-W}}$	28.83	0.51	0.30	28.38	0.32	0.82	4.97	0.32	0.30
	(17.54)	(0.07)	(0.05)	(22.32)	(0.05)	(0.15)	(3.28)	(0.06)	(0.04)
True	30	0.5	0.8	30	1.8	0.8	5	1	0.8
θ_{CI}	33.74	0.52	0.81	29.92	1.71	0.78	5.79	1.18	0.85
	(19.22)	(0.17)	(0.19)	(4.87)	(0.26)	(0.15)	(2.21)	(0.29)	(0.08)
$\nu_{F\text{-CI}}$	34.61	0.52	0.80	31.49	1.78	0.77	5.59	1.04	0.76
	(18.49)	(0.09)	(0.17)	(3.77)	(0.16)	(0.12)	(1.23)	(0.17)	(0.06)
$\ell_{2\text{-C}}$	29.54	0.57	0.72	29.94	1.76	0.78	6.04	1.13	0.75
	(23.69)	(0.28)	(0.35)	(4.44)	(0.20)	(0.13)	(2.82)	(0.31)	(0.40)
$\ell_{2\text{-CI}}$	29.66	0.52	0.82	30.50	1.80	0.79	5.26	1.04	0.78
	(17.10)	(0.07)	(0.18)	(3.96)	(0.11)	(0.13)	(1.01)	(0.13)	(0.07)
$\ell_{3\text{-W}}$	29.39	0.51	0.81	30.48	1.80	0.79	5.24	1.04	0.78
	(16.78)	(0.07)	(0.14)	(3.87)	(0.09)	(0.12)	(0.92)	(0.12)	(0.07)
True	30	0.5	1.5	30	1.8	1.5	15	1	0.8
θ_{CI}	33.52	0.51	1.56	29.73	1.72	1.53	15.28	1.06	0.82
	(17.60)	(0.15)	(0.43)	(4.30)	(0.25)	(0.37)	(4.89)	(0.29)	(0.15)
$\nu_{F\text{-CI}}$	35.12	0.51	1.54	31.40	1.79	1.49	16.34	1.04	0.78
	(16.58)	(0.08)	(0.39)	(3.45)	(0.15)	(0.33)	(3.35)	(0.16)	(0.12)
$\ell_{2\text{-C}}$	29.10	0.63	1.50	29.16	1.74	1.48	15.52	1.08	0.78
	(29.14)	(0.45)	(0.41)	(7.00)	(0.27)	(0.26)	(5.19)	(0.29)	(0.24)
$\ell_{2\text{-CI}}$	28.94	0.51	1.57	30.30	1.80	1.52	15.09	1.03	0.80
	(15.60)	(0.06)	(0.42)	(3.70)	(0.10)	(0.33)	(3.49)	(0.12)	(0.13)
$\ell_{3\text{-W}}$	28.83	0.51	1.55	30.28	1.80	1.51	15.07	1.02	0.79
	(15.56)	(0.06)	(0.31)	(3.61)	(0.08)	(0.25)	(3.44)	(0.11)	(0.11)

5·2. *Trivariate case*

We simulated $T = 30$ independent realizations from a trivariate Hüsler–Reiss process at 15 random locations uniformly generated from $[0, 100]^2$. The true values of the parameters were $\alpha = 30$, $\kappa = 0.5$, $\lambda_{12} = 0.3$, and $\lambda_{13} = \lambda_{23} = 1.5$. We repeated this simulation 1000 times to calculate empirical standard errors and mean squared errors of the parameter estimates. The model parameters were estimated using the pairwise, triplewise and the weighted triplewise likelihoods. Specifically, we considered two types of pairwise composite likelihood approaches, $\ell_{2\text{-CI}}$ and $\ell_{2\text{-C}}$, defined similarly as in the first simulation. The former included 990 pairs and the latter included 630 pairs. We also considered two types of triplewise composite likelihoods based on all possible triples, denoted as $\ell_{3\text{-CI}}$, and the composite likelihood based on only cross triples, see the restricted set D_3^*, denoted as $\ell_{3\text{-C}}$. Each included 4110 triples and 2740 triples, respectively. The results of the new composite likelihood (12) approach, $\ell_{3\text{-W}}$, were also reported. It included 1260 quadruples. Figure 1 shows the boxplots of the 1000 independent estimates of the range

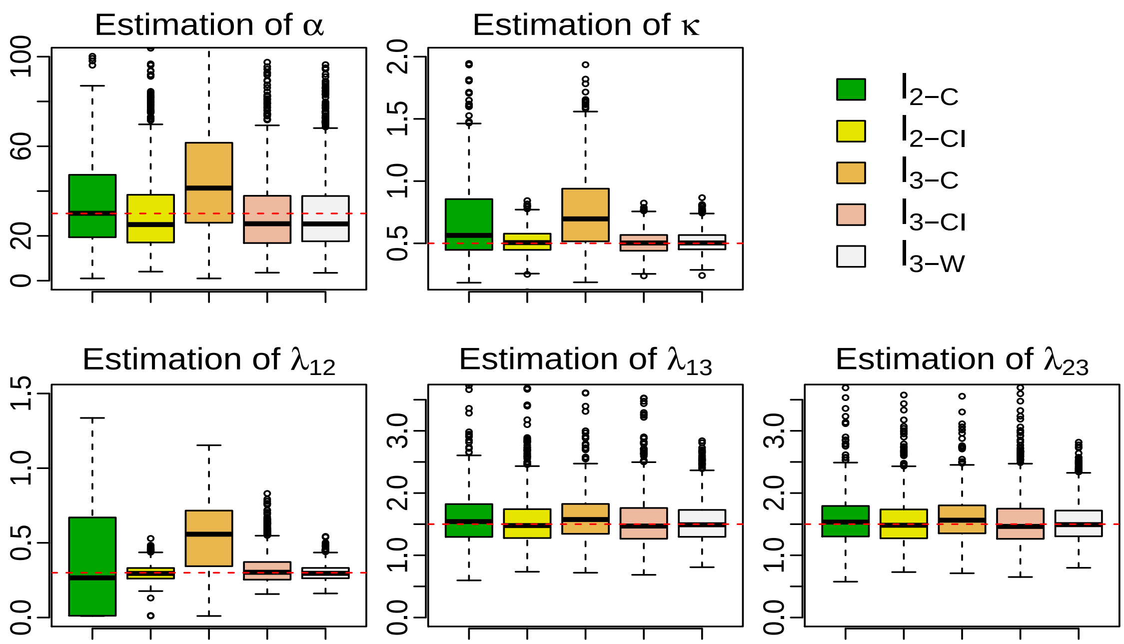

Fig. 1. Boxplots of the 1000 independent estimates of the range parameter ϕ, smoothness parameter κ, and cross-variable dependence parameters $(\lambda_{12}, \lambda_{13}, \lambda_{23})$, in which the true parameter values are $\alpha = 30$, $\kappa = 0.5$, $\lambda_{12} = 0.3$, $\lambda_{13} = \lambda_{23} = 1.5$. In each subfigure boxes from left to right correspond to composite likelihood estimates based on: $\ell_{2\text{-C}}$, $\ell_{2\text{-CI}}$, $\ell_{3\text{-C}}$, $\ell_{3\text{-CI}}$, and $\ell_{3\text{-W}}$.

parameter, ϕ, the smoothness parameter, κ, and the cross-correlation parameters, $(\lambda_{12}, \lambda_{13}, \lambda_{23})$. Overall, all the considered composite likelihood approaches produced fairly reasonable estimates of the model parameters. Similarly to our findings in the first simulation study, we see that both the $\ell_{3\text{-CI}}$ and the $\ell_{2\text{-CI}}$ estimators have reduced standard errors compared to those of the $\ell_{3\text{-C}}$ and the $\ell_{2\text{-C}}$ estimators, especially for the range parameter and the smoothness parameter. This finding suggests the need to include within-variable and cross-variable-same-location pairs or triples in parameter estimation. The new composite likelihood estimator, $\ell_{3\text{-W}}$, is the best among all the composite likelihood estimators considered, especially when assessing strong dependence levels.

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