Received: 14 May 2017 | Revised: 25 December 2017 | Accepted: 23 January 2018

DOI: 10.1111/sjos.12325

ORIGINAL ARTICLE

Scandinavian Journal of Statistics

Linear factor copula models and their properties

Pavel Krupskii | Marc G. Genton

Statistics Program, King Abdullah University of Science and Technology

Correspondence
Pavel Krupskii, Statistics Program, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.
Email: pavel.krupskiy@kaust.edu.sa

Abstract

We consider a special case of factor copula models with additive common factors and independent components. These models are flexible and parsimonious with $O(d)$ parameters where d is the dimension. The linear structure allows one to obtain closed form expressions for some copulas and their extreme-value limits. These copulas can be used to model data with strong tail dependencies, such as extreme data. We study the dependence properties of these linear factor copula models and derive the corresponding limiting extreme-value copulas with a factor structure. We show how parameter estimates can be obtained for these copulas and apply one of these copulas to analyse a financial data set.

KEYWORDS

copula, extreme-value copula, heavy tails, linear factor structure, tail asymmetry

1 | INTRODUCTION

Modelling multivariate data with complex dependence structures is a challenging task. Models based on multivariate normality might not be suitable when tail dependence or asymmetric dependence is found in the data. Copulas, on the other hand, are capable of modelling multivariate data with tail dependence or asymmetric dependence and have, therefore, gained popularity as useful tools for constructing flexible multivariate distributions. A copula is a multivariate cumulative distribution function (cdf) with uniform $U(0,1)$ marginals. Sklar (1959) showed that, for any continuous multivariate cdf (F_d) with univariate marginal cdfs, $F_1^1, \dots, F_1^d$, there exists a unique copula (C_d) such that $F_d(z_1, \dots, z_d) = C_d\{F_1^1(z_1), \dots, F_1^d(z_d)\}$ for any $z_1, \dots, z_d$. This copula function allows a multivariate distribution to be constructed from the given marginal cdfs.

Copulas are used in many applications to model non-Gaussian data, such as financial data (Patton, 2006), hydrology data (Genest & Favre, 2007), spatial data (Bárdossy & Li, 2008), and others. One special class of copula models is that with a factor structure (factor copula models). These parsimonious and flexible models can be used for modelling data where there exist one or several unobserved (latent) factors that affect the joint dependence among all of the variables. These factor copula models are used to model financial data (Krupskii & Joe, 2013; 2015a; Oh & Patton, 2017), item response data (Nikoloulopoulos & Joe, 2015), and spatial data (Krupskii, Huser, & Genton, 2018). Unlike the vine copula models (Aas, Czado, Frigessi, & Bakken, 2009; Kurowicka & Cooke, 2006), in which the dependence structure is selected based on the likelihood, the models with a factor structure can be nicely interpreted. One example is a credit portfolio when some severe economic shocks can affect all the portfolio components, leading to multiple defaults. These shocks usually cannot be easily measured, and there might be many different factors contributing to these shocks, so it is natural to assume that these are unobserved variables.

In this paper, we study the special class of factor copula models with linear structures, as proposed by Krupskii & Joe (2013). These copulas can handle a wide range of dependencies and have $O(d)$ parameters. To construct a linear factor copula, we use a linear combination of common factors and independent factors, each having the same distribution. Similar ideas were used in the construction of generalized Archimedean copulas (Rogge & Schönbucher, 2003) and models based on comonotonic factors (Hua & Joe, 2017). Linear factor copula models are a special case of the latter models with linear loadings. The limiting extreme-value copulas (Gudendorf & Segers, 2010) with factor structures can be derived for this class of models, in closed form in some cases, and the parameters can be efficiently estimated using a composite maximum likelihood approach (for continuous copulas) or the method of moments (for copulas with singular components).

The rest of the paper is organized as follows. In Section 2, we introduce the class of one-factor copula models with linear structures and study their dependence properties. We derive the limiting extreme-value copulas for this class of models and further extend this approach to models with $p \geq 2$ factors in Section 3. As a special case of this class of linear factor copula models, we introduce an extension of the Marshall–Olkin copula (Marshall & Olkin, 1967) with $p \geq 2$ factors affecting all of the components in a system. In Section 4, we show how the copula parameter estimates can be obtained for the models introduced in the two previous sections. In Section 5, we apply one of the proposed linear factor copula models to a financial data set, and Section 6 concludes with a discussion.

2 | ONE-FACTOR COPULA MODEL WITH A LINEAR STRUCTURE

We assume that $\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_d \sim_{\text{i.i.d.}} F_{\mathcal{E}}$, where $F_{\mathcal{E}}$ is a cdf with a positive density on $\mathbb{R}_+$. We define the variables

$$W_j = \alpha_j \mathcal{E}_0 + \mathcal{E}_j, \quad \alpha_j \geq 0 \text{ and } j = 1, \dots, d. \tag{1}$$

This is a special case of the one-factor copula models (Krupskii & Joe, 2013). Let $F_d^{\mathbf{W}}$ and $C_d^{\mathbf{W}}$ be the multivariate cdf and the copula, respectively, corresponding to the joint distribution of the vector $\mathbf{W} = (W_1, \dots, W_d)^\top$, and let $C_d^{\mathbf{W}}$ be the corresponding extreme-value copula. We use small letters to denote the corresponding probability density functions (pdf). It follows that

$$\begin{aligned} C_d^{\mathbf{W}}(u_1, \dots, u_d) &= F_d^{\mathbf{W}}\{(F_1^{W_1})^{-1}(u_1), \dots, (F_1^{W_d})^{-1}(u_d)\}, \\ c_d^{\mathbf{W}}(u_1, \dots, u_d) &= \frac{f_d^{\mathbf{W}}\{(F_1^{W_1})^{-1}(u_1), \dots, (F_1^{W_d})^{-1}(u_d)\}}{\prod_{j=1}^d f_1^{W_j}\{(F_1^{W_j})^{-1}(u_j)\}}, \end{aligned} \tag{2}$$

where $F_1^{W_j}$ and $f_1^{W_j}$ are the cdf and pdf, respectively, of $W_j, j = 1, \dots, d$.

Here, we focus on the upper tail properties of $C_d^{\mathbf{W}}$; the lower tail properties can be considered analogously. We let $\lambda_U^{j,k}$ be the upper tail dependence coefficient for the copula corresponding to the distribution of (W_j, W_k), $1 \le j < k \le d$. In order to generate upper tail dependence in model (1), so that $\lambda_U^{j,k} > 0$ for different pairs (W_j, W_k), one needs to use an $F_{\mathcal{E}}$ with heavy tails, as the next proposition shows.

Proposition 1. *Let $\bar{F}^{-1}(q)$ be the upper q-quantile of the cdf F. Assume that for any $\alpha_j > 0$, $j = 1, \dots, d$, there exists some $\delta > 1$ such that $(\bar{F}_1^{W_j})^{-1}(q) \ge \delta\alpha_j\bar{F}_{\mathcal{E}}^{-1}(q)$, and that for any $\delta > 1$, $\bar{F}_{\mathcal{E}}\{\delta\bar{F}_{\mathcal{E}}^{-1}(q)\} = o(q)$ as $q \to 0$. It follows that $\lambda_U^{j,k} = 0$ for any $1 \le j < k \le d$.*

The proof is given in Appendix A.1.

Remark 1. In Appendix A.1, we also show that the conditions of Proposition 1 are satisfied for the Weibull distribution $F_{\mathcal{E}}(x) = 1 - \exp(-x^{-\gamma})$, $x > 0$, with $\gamma > 1$.

However, not all heavy-tailed distributions $F_{\mathcal{E}}$ generate flexible dependence structures. If $F_{\mathcal{E}}$ is a subexponential distribution with no regularly varying upper tail, then each pair of variables in (1) either has no tail dependence or has comonotonic tail dependence, with the upper tail dependence coefficient being equal to one.

Proposition 2. *Assume that $F_{\mathcal{E}}$ belongs to the class of subexponential distributions defined by Chistyakov (1964), and that for $\kappa > 1$, $\bar{F}_{\mathcal{E}}(\kappa x) = o\{\bar{F}_{\mathcal{E}}(x)\}$ as $x \to \infty$. It follows that $\lambda_U^{j,k} = 0$ for any $1 \le j < k \le d$, such that* $\min\{\alpha_j, \alpha_k\} \le 1$*; and $\lambda_U^{j,k} = 1$ for any $1 \le j < k \le d$, such that* $\min\{\alpha_j, \alpha_k\} > 1$.

Proposition 2 is a special case of the more general result with $p \ge 1$ common factors; see Proposition 5 in Section 3. Examples of distributions that satisfy the conditions of Proposition 2 include the Weibull distribution with $F_{\mathcal{E}}(x) = 1 - \exp(\gamma_1 x^{-\gamma_2})$ where $\gamma_1 > 0$ and $0 < \gamma_2 < 1$, or the modified exponential distribution $F_{\mathcal{E}}(x) = 1 - \exp\{-x(\ln x)^{-\gamma_0}\}$ where $\gamma_0 > 0$.

One interesting copula family arises in the boundary case, where $F_{\mathcal{E}}$ is the exponential distribution: $F_{\mathcal{E}}(w) = 1 - \exp(-w)$ for $w > 0$. This is not a subexponential distribution; the copula $C_d^{\mathbf{W}}$ allows the generation of flexible dependence structures depending on the choice of parameters $\alpha_1, \dots, \alpha_d$; for example, the tail dependence can be obtained for some choice of parameters $\alpha_1, \dots, \alpha_d$. For $w_1, \dots, w_d > 0$,

$$
\begin{aligned}
F_d^{\mathbf{W}}(w_1, \dots, w_d) &= \int_0^{\min_j(w_j/\alpha_j)} \prod_{j=1}^{d} \{1 - \exp(\alpha_j w_0 - w_j)\} \exp(-w_0) \mathrm{d}w_0, \\
f_d^{\mathbf{W}}(w_1, \dots, w_d) &= \int_0^{\min_j(w_j/\alpha_j)} \exp\left(w_0 \sum_{j=1}^{d} \alpha_j - \sum_{j=1}^{d} w_j \right) \exp(-w_0) \mathrm{d}w_0 \\
&= \frac{\exp(-\sum_{j=1}^{d} w_j)}{\sum_{j=1}^{d} \alpha_j - 1} \left[\exp\left\{ \min_j(w_j/\alpha_j) \left(\sum_{j=1}^{d} \alpha_j - 1 \right) \right\} - 1 \right].
\end{aligned} \tag{3}
$$

The marginal cdf and pdf, $F_1^{W_j}$ and $f_1^{W_j}$, are

$$
F_1^{W_j}(w_j) = 1 - \frac{\alpha_j \exp(-w_j/\alpha_j) - \exp(-w_j)}{\alpha_j - 1} \text{ and } f_1^{W_j}(w_j) = \frac{\exp(-w_j/\alpha_j) - \exp(-w_j)}{\alpha_j - 1},
$$

so the copula density, $c_d^{\mathbf{W}}$, can be obtained in closed form from (2). In the general case, one-dimensional integration is required to obtain the copula density in one-factor copula models; see Krupskii & Joe (2013). This copula does not require numerical integration, and therefore, the parameters of this copula can be estimated very quickly using the maximum likelihood approach, even if d is very large; more details are given in Section 4.

Also, the limiting extreme-value copula, $C_d^{\mathbf{W}}$, can be obtained in a simple form in this case, as the next proposition shows.

Proposition 3. *In model (1), let $F_{\mathcal{E}}(x) = 1 - \exp(-x)$ for $x > 0$. For any $1 \le j < k \le d$ such that $\min(\alpha_j, \alpha_k) \le 1$, $\lambda_U^{j,k} = 0$. For $\alpha_1, \ldots, \alpha_d > 1$, the stable upper tail dependence function (Segers, 2012) for $F_d^{\mathbf{W}}$ is*

$$\ell_d(x_1, \ldots, x_d) = y_{(1)} \left[1 - \sum_{m=1}^{d} (-1)^m \sum_{1 \le j_1 < \cdots < j_m \le d} \frac{1}{\sum_{l=1}^{m} \alpha_{j_l} - 1} \prod_{l=1}^{m} \left\{ \frac{y_{j_l}}{y_{(1)}} \right\}^{\alpha_j} \right],$$

where $y_j = (1 - 1/\alpha_j)x_j$, $j = 1, \ldots, d$, and $y_{(1)} = \max_j y_j$.

The proof of this proposition is given in Appendix A.2. Proposition 3 implies that $C_d^{\mathbf{W}}(u_1, \ldots, u_d) = \exp\{-\ell_d(-\ln u_1, \ldots, -\ln u_d)\}$. The function $\ell_d(x_1, \ldots, x_d)$ requires numerical integration in the general case of one-factor copula models; see Joe (2014). Therefore, parameter estimation can be computationally demanding for a large d. However, because the stable upper tail dependence function in Proposition 3 is given in closed form, the composite likelihood approach can be used to efficiently estimate the parameters of $C_d^{\mathbf{W}}$ for any d; see, for example, Lindsay (1998) and Cox & Reid (2004). We provide more details in Section 4.

For this extreme-value copula, $\ell_d(x_1, \ldots, x_d)$ is a permutation asymmetric function unless $\alpha_1 = \cdots = \alpha_d$; therefore, $C_d^{\mathbf{W}}$ is permutation asymmetric. Also, with $\alpha_j > 1$ and $\alpha_k > 1$,

$$\lambda_U^{j,k} = 1 - \frac{1}{\alpha_j + \alpha_k - 1} \left\{ \frac{\alpha_{(2)} - 1}{\alpha_{(1)} - 1} \right\}^{\alpha_{(2)}-1} \left\{ \frac{\alpha_{(1)}}{\alpha_{(2)}} \right\}^{\alpha_{(2)}},$$

where $\alpha_{(1)} = \max(\alpha_j, \alpha_k)$ and $\alpha_{(2)} = \min(\alpha_j, \alpha_k)$. $\lambda_U^{j,k} \to 1$ if $\alpha_j \to \infty$ and $\alpha_k \to \infty$.

Now, consider the bivariate marginal copula $C_2^{\mathbf{W}}$. Permutation asymmetry of the copula $C_2^{\mathbf{W}}$ is stronger when the difference $|\alpha_1 - \alpha_2|$ is larger. Figure 1 shows scatter plots with sample size $N = 2000$ from this copula with different parameters α_1 and α_2, such that the upper tail dependence coefficient $\lambda_U^{1,2} = 0.5$.

The original copula $C_d^{\mathbf{W}}$ can also generate a wide range of dependencies, including tail independence, when some or all parameters $\alpha_1, \ldots, \alpha_d$ are less than one. At the same time, the tail independence for many multivariate tail dependent copulas proposed in the literature can only be obtained on the boundary of the parameter space; see Joe (2014).

Different dependence structures can be obtained for heavy-tailed distributions $F_{\mathcal{E}}$ with regularly varying upper tails. The limiting extreme-value copula corresponding to $C_d^{\mathbf{W}}$ has a singular component in this case.

Proposition 4. *Assume that $F_{\mathcal{E}}(x) = 1 - x^{-\beta}$, where $x \ge 1$. Let $\theta_j = \alpha_j^\beta/(1 + \alpha_j^\beta)$, $j = 1, \ldots, d$. Then the limiting extreme-value copula corresponding to $C_d^{\mathbf{W}}$ is the Marshall–Olkin copula (Marshall & Olkin, 1967) with a one-factor structure,*

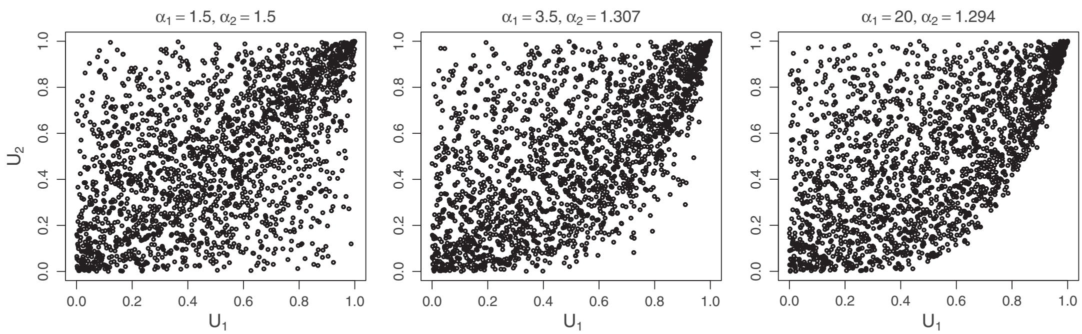

FIGURE 1 Scatter plots of data sets generated from the copula $C_2^{\mathbf{W}}$ from Proposition 3 with $\alpha_1 = \alpha_2 = 1.5$ (left), $\alpha_1 = 3.5, \alpha_2 = 1.307$ (middle) and $\alpha_1 = 20, \alpha_2 = 1.294$ (right); $\lambda_U^{1,2} = 0.5$ in all cases. The sample size is 2,000

$$C_d^{\mathbf{W}}(u_1, \dots, u_d) = \left(\prod_{j=1}^{d} u_j^{1-\theta_j}\right) \min_j \left(u_j^{\theta_j}\right).$$

Proposition 4 is a special case of the more general result with $p \geq 1$ common factors; see Proposition 7 in Section 3. This copula has a singular component $u_1^{1-\theta_1} = \cdots = u_d^{1-\theta_d}$ and parsimonious dependence structure with d parameters $\theta_1, \dots, \theta_d$. Similar copulas are studied in Mazo, Girard, and Forbes (2016); see also Cherubini, Durante, and Mulinacci (2015) for the general construction principle for copulas of this type. This copula can be used in reliability theory when all of the components in a system can be simultaneously affected by a single shock. In the next section, we introduce an extension to this copula with a p-factor structure for $p \geq 2$.

3 | *p*-FACTOR STRUCTURES WITH $p \geq 2$

More general models with p factors can be obtained by using p independent common factors:

$$W_j = \sum_{s=1}^{p} \alpha_{js} \mathcal{E}_{0s} + \mathcal{E}_j, \quad \alpha_{js} \geq 0, \quad j = 1, \dots, d, \quad s = 1, \dots, p, \tag{4}$$

where $\mathcal{E}_{01}, \dots, \mathcal{E}_{0p}, \mathcal{E}_1, \dots, \mathcal{E}_d \sim_{i.i.d.} F_{\mathcal{E}}$, and $F_{\mathcal{E}}$ is a continuous cdf on $\mathbb{R}_+$, as in Section 2. Most of the results from Section 2 also hold for $p \geq 2$, with small modifications.

Proposition 5. *Under the assumptions of Proposition 2, if we have* $\alpha_{js_0} = \max_s \alpha_{js} > 1$ *and* $\alpha_{ks_0} = \max_s \alpha_{ks} > 1$ *for some* $1 \leq s_0 \leq p$, *then* $\lambda_U^{j,k} = 1$; *otherwise,* $\lambda_U^{j,k} = 0$.

The proof of this proposition is given in Appendix A.3.

Because subexponential distributions without regularly varying tails are not suitable for the generation of flexible dependence structures, even when using $p \geq 2$ linear factors, we consider the exponential distribution $F_{\mathcal{E}}(w) = 1 - \exp(-w)$, when $w > 0$. Computations become very complicated in the general case; therefore, here, we focus on $p = 2$. In fact, even one factor is usually sufficient to adequately describe the dependence structure; we demonstrate this in Section 5 by fitting model (4) with $p = 1$ to simulated data sets with $p \geq 1$ common factors. Although it is difficult to obtain a simple formula for the limiting extreme-value copula in the general case of $d \geq 2$, it is possible to obtain a closed-form formula for the stable upper tail dependence function, $\ell_d(x_1, \dots, x_d)$, for each bivariate margin. Without loss of generality, we consider the copula corresponding to the distribution of the first two variables, W_1 and W_2. The next proposition gives the formula for $\ell_2(x_1, x_2)$.

Proposition 6. *In model (4), let $p = 2$ and $F_{\mathcal{E}}(x) = 1 - \exp(-x)$ where $x > 0$. It follows that $\lambda_U^{1,2} = 0$ if $\min(\alpha_{11}, \alpha_{21}) < 1$ and $\min(\alpha_{12}, \alpha_{22}) < 1$, or if $(\alpha_{11} - \alpha_{12})(\alpha_{21} - \alpha_{22}) \leq 0$. Assume here that $\alpha_{11} > \max(\alpha_{12}, 1)$ and $\alpha_{21} > \max(\alpha_{22}, 1)$. Let $y_j = m_j x_j$ where $m_j = (1 - 1/\alpha_{j1})(1 - \alpha_{j2}/\alpha_{j1})$, $j = 1, 2$. Define*

$$\varphi(y_1, y_2) = (m_2^{-1} - m_1^{-1} + k_2 - k_1) y_1^{-\frac{1}{\psi}\left(1 - \frac{\alpha_{22}}{\alpha_{21}}\right)} y_2^{\frac{1}{\psi}\left(1 - \frac{\alpha_{12}}{\alpha_{11}}\right)},$$

where $\psi = a_{22}/a_{21} - a_{12}/a_{11}$, $k_1 = \alpha_{11}^2[(\alpha_{11} + \alpha_{21} - 1)(\alpha_{21} - 1)\{\alpha_{11}(1 - \alpha_{22}) - \alpha_{12}(1 - \alpha_{21})\}]^{-1}$ and $k_2 = \alpha_{21}^2[(\alpha_{11} + \alpha_{21} - 1)(\alpha_{11} - 1)\{\alpha_{21}(1 - \alpha_{12}) - \alpha_{22}(1 - \alpha_{11})\}]^{-1}$. The stable upper tail dependence function for $F_2^{\mathbf{W}}$ is

$$\ell_2(x_1, x_2) = \begin{cases} x_1 + k_1 y_1^{1-\alpha_{21}} y_2^{\alpha_{21}} + \varphi(y_1, y_2), & \psi > 0, y_1 > y_2; \\ x_2 + k_2 y_1^{\alpha_{11}} y_2^{1-\alpha_{11}}, & \psi > 0, y_1 \leq y_2; \\ x_1 + k_1 y_1^{1-\alpha_{21}} y_2^{\alpha_{21}}, & \psi \leq 0, y_1 > y_2; \\ x_2 + k_2 y_1^{\alpha_{11}} y_2^{1-\alpha_{11}} - \varphi(y_1, y_2), & \psi \leq 0, y_1 \leq y_2. \end{cases}$$

The proof is given in Appendix A.4. The formula for $\ell_2(x_1, x_2)$ can be used to obtain copula parameter estimates using the pairwise likelihood, similar to the case of $p = 1$. More details are given in Section 4.

Remark 2. Model (4) with $p = 2$ exponential factors is not identifiable in the general case. Let $\alpha_{21} - \alpha_{22} = \alpha_1^*(\alpha_{11} - \alpha_{12})$. As it follows from Proposition 6, $\ell_2(x_1, x_2)$ only depends on α_{11}, α_{21}, and α_1^* so that four parameters are redundant. With $d \geq 2$, the bivariate marginals depend on d parameters α_{j1}, $j = 1, \ldots, d$, and $d - 1$ parameters α_j^*, $1 \leq j \leq d - 1$, where $\alpha_{j+1,1} - \alpha_{j+1,2} = \alpha_j^*(\alpha_{11} - \alpha_{12})$. To estimate the parameters in this model, one can set $\alpha_{12} = 1$ and estimate the remaining parameters using the pairwise likelihood.

We now introduce one extension of the Marshall–Olkin copula, which is obtained as an extreme-value limit of $C_d^{\mathbf{W}}$ that links $W_1, \ldots, W_d$, as defined in (4), with $F_{\mathcal{E}}(x)$ having a regularly varying upper tail.

Proposition 7. *Assume that $\bar{F}_{\mathcal{E}}(x) \sim_{x\to\infty} x^{-\beta}$. Let $\theta_{js} = \alpha_{js}^{\beta}/(1 + \sum_{s=1}^{p} \alpha_{js}^{\beta})$, $j = 1, \ldots, d$, $s = 1, \ldots, p$. The limiting extreme-value copula corresponding to $C_d^{\mathbf{W}}$ is the (extended) Marshall–Olkin copula with a p-factor structure:*

$$C_d^{\mathbf{W}}(u_1, \ldots, u_d) = \left(\prod_{j=1}^{d} u_j^{1 - \sum_{s=1}^{p} \theta_{js}}\right) \prod_{s=1}^{p} \left\{\min_j \left(u_j^{\theta_{js}}\right)\right\}. \tag{5}$$

The proof of this proposition is given in Appendix A.5. The copula $C_d^{\mathbf{W}}$ has a parsimonious dependence structure with pd parameters and the singular components $\{(u_1, \ldots, u_d)^\top : u_j^{\theta_{js}} = u_k^{\theta_{ks}}, 1 \leq j < k \leq d, \quad s = 1, \ldots, p\}$. It follows that $\lambda_U^{j,k} = \sum_{s=1}^{p}\{\theta_{js} + \theta_{ks} - \max(\theta_{js}, \theta_{ks})\} = \sum_{s=1}^{p} \min(\theta_{js}, \theta_{ks})$. Figure 2 shows the data sets generated from this copula with $d = 2$ and $p = 1, 2, 3$, and parameters selected such that $\lambda_U^{1,2} \approx 0.45$ in all cases.

Remark 3. Unlike model (4) with exponential factors and $p = 2$, the parameters θ_{js}, $j = 1, \ldots, d, s = 1, \ldots, p$, can all be identified in the general case $p \geq 1$.

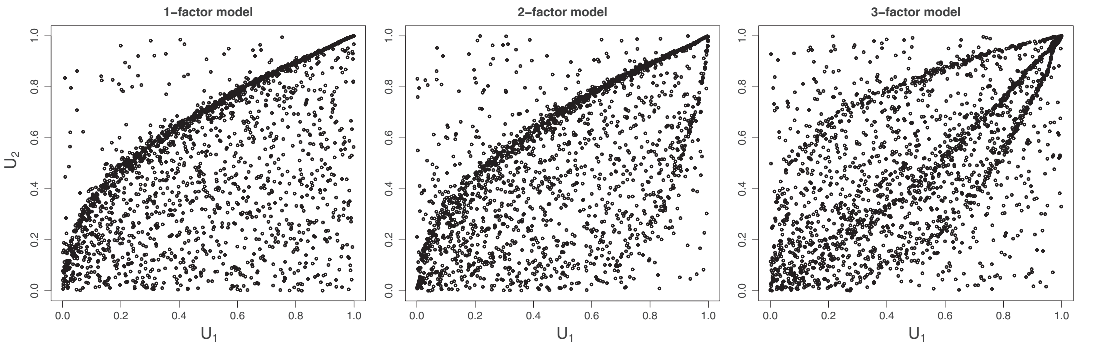

FIGURE 2 Scatter plots of data sets generated from copula $C_2^{\mathbf{W}}$ from Proposition 7 with $p = 1$ and $\alpha_{11} = 0.95$, $\alpha_{21} = 2$ (left); $p = 2$, and $\alpha_{11} = 1.2, \alpha_{12} = 1.2, \alpha_{21} = 1.6, \alpha_{22} = 0.8$ (middle); $p = 3$ and $\alpha_{11} = 1.1, \alpha_{12} = 0.8, \alpha_{13} = 1$, $\alpha_{21} = 0.9, \alpha_{22} = 1.2, \alpha_{23} = 1$ (right); $\lambda_U^{1,2} \approx 0.45$ in all cases. The sample size is 2,000

The first data set has one singular component, $u_2 = u_1^{2.1}$; the second data set has two singular components, $u_1 = u_2^{0.12}$ and $u_1 = u_2^{2.04}$; and the third data set has three singular components, $u_1 = u_2^{0.37}$, $u_1 = u_2^{0.82}$, and $u_1 = u_2^{4.15}$. This extended Marshall–Olkin copula can be useful for modelling the lifetime of a system whose components can all be affected simultaneously by p different shocks.

We now give the formula of the Spearman's rho for the $(1, 2)$th bivariate margin.

Proposition 8. *Let* $(U_1, U_2) \sim C_2^{\mathbf{W}}$ *where* $C_2^{\mathbf{W}}$ *is the copula from Proposition 7 with* $d = 2$. *Define* $\zeta_s = \theta_{2s}/\theta_{1s}$ *and let* $\zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_p$, *with* $\zeta_0 := \infty$ *and* $\zeta_{p+1} := 0$. *The Spearman's rho is* $\mathrm{cor}(U_1, U_2) = 12\rho_{12} - 3$ *where*

$$\rho_{12} = \sum_{s=0}^{p} \frac{\zeta_s - \zeta_{s+1}}{(\zeta_{s+1}\Theta_{1s} + \Theta_{2s})(\zeta_s\Theta_{1s} + \Theta_{2s})}, \quad \Theta_{1s} = 2 - \sum_{i=1}^{s}\theta_{1i}, \quad \Theta_{2s} = 2 - \sum_{i=s+1}^{p}\theta_{2i}.$$

The proof of this proposition is given in Appendix A.6. This formula can be useful to obtain the copula parameter estimates; see more details in Section 4.

4 | PARAMETER ESTIMATION

In this section, we provide more details about estimating the parameters of factor copula models (1) and (4), with the exponential and Pareto factors, and of their limiting extreme-value copulas. Throughout this section, we assume that $\mathbf{u} = (\mathbf{u}_1^\top, \ldots, \mathbf{u}_d^\top)^\top = \{(u_{i1}, \ldots, u_{id})^\top\}_{i=1}^N$ is a sample of size N from the corresponding copula, either $C_d^{\mathbf{W}}$ or $C_d^{\mathbf{W}}$, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)^\top$ is a vector of dependence parameters. Nonparametric ranks can be used to transform data with non-uniform univariate marginals to the $(0, 1)$ scale.

4.1 | Likelihood maximization for $C_d^{\mathbf{W}}$ with $F_{\mathcal{E}}$ being the exponential distribution

From (2) and (3), the log-likelihood for $C_d^{\mathbf{W}}$ in the model (1) is

$$l_d(\mathbf{u};\boldsymbol{\alpha}) = \sum_{i=1}^{N} \ln \left| \exp\left\{ \min_j(w_{ij}/\alpha_j) \left(\sum_{j=1}^{d} \alpha_j - 1 \right) \right\} - 1 \right|$$
$$- \sum_{i,j=1}^{N,d} \ln \left| \exp\{(1-1/\alpha_j)w_{ij}\} - 1 \right| + N \sum_{j=1}^{d} \ln|\alpha_j - 1| - N \ln \left| \sum_{j=1}^{d} \alpha_j - 1 \right|,$$

where $w_{ij} = (F_1^{W_j})^{-1}(u_{ij};\alpha_j)$, $i = 1, \ldots, N, j = 1, \ldots, d$. We find the gradient of $l_d(\mathbf{u};\boldsymbol{\alpha})$, in order to use it in an optimization algorithm. Let $M_i := \min_j(w_{ij}/\alpha_j)$ and $\tilde{\alpha} = \sum_{j=1}^{d} \alpha_j - 1$. For $k = 1, \ldots, d$,

$$\frac{\partial l_d(\mathbf{u};\boldsymbol{\alpha})}{\partial \alpha_k} = \sum_{i=1}^{N} \frac{\{M_i + (\partial M_i/\partial\alpha_k)\cdot\tilde{\alpha} I(M_i = w_{ik}/\alpha_k)\}\exp(M_i\tilde{\alpha})}{\exp(M_i\tilde{\alpha}) - 1}$$
$$- \sum_{i=1}^{N} \frac{w_{ik}/\alpha_k^2 + (\partial w_{ik}/\partial\alpha_k)(1-1/\alpha_k)}{\exp\{(1-1/\alpha_k)w_{ik}\} - 1} + \frac{N}{\alpha_k - 1} - \frac{N}{\tilde{\alpha}},$$

where $\partial M_i/\partial\alpha_k = (\partial w_{ik}/\partial\alpha_k)/\alpha_k - w_{ik}/\alpha_k^2$ and

$$\partial w_{ik}/\partial\alpha_k = -\frac{\partial F_1^{W_k}(w_{ik};\alpha_k)/\partial\alpha_k}{f_1^{W_k}(w_{ik};\alpha_k)} = \frac{1}{1-\alpha_k} + \frac{w_{ik}/\alpha_k}{1-\exp\{(1/\alpha_k - 1)w_{ik}\}}.$$

Then the log-likelihood, $l_d(\mathbf{u};\boldsymbol{\alpha})$, and its gradient, $\partial l_d(\mathbf{u};\boldsymbol{\alpha})/\partial\boldsymbol{\alpha}$, can be used in the Newton–Raphson algorithm to obtain the parameter estimates.

4.2 | Pairwise likelihood estimation for $C_d^{\mathbf{W}}$ with $F_\mathcal{E}$ being the exponential distribution

With $p = 1$, the pairwise log-likelihood for $C_d^{\mathbf{W}}$ in the model (1) can be written as

$$l_d^2(\mathbf{u};\boldsymbol{\alpha}) = \sum_{1\le k<j\le d} l_{j,k}(\mathbf{u}_j,\mathbf{u}_k;\alpha_j,\alpha_k),$$

where

$$l_{j,k}(\mathbf{u}_j,\mathbf{u}_k;\alpha_j,\alpha_k) = -\sum_{i=1}^{N} \ell_2(\tilde{u}_{ij},\tilde{u}_{ik};\alpha_j,\alpha_k)\ln\zeta(\tilde{u}_{ij},\tilde{u}_{ik};\alpha_j,\alpha_k),$$

$\tilde{u}_{ij} = -\ln u_{ij}$, $i = 1, \ldots, N, j = 1, \ldots, d$, and

$$\zeta(\tilde{u}_{ij},\tilde{u}_{ik};\alpha_j,\alpha_k) = \frac{\partial\ell_2(\tilde{u}_{ij},\tilde{u}_{ik};\alpha_j,\alpha_k)}{\partial u_{ij}} \cdot \frac{\partial\ell_2(\tilde{u}_{ij},\tilde{u}_{ik};\alpha_j,\alpha_k)}{\partial u_{ik}} - \frac{\partial^2\ell_2(\tilde{u}_{ij},\tilde{u}_{ik};\alpha_j,\alpha_k)}{\partial u_{ij}\partial u_{ik}}.$$

Without loss of generality, we assume $d = 2$. We define

$$\vartheta(\alpha_1,\alpha_2) = \frac{1}{\alpha_1+\alpha_2-1}\left(\frac{\alpha_2-1}{\alpha_1-1}\right)^{\alpha_2-1}\left(\frac{\alpha_1}{\alpha_2}\right)^{\alpha_2}.$$

Then

$$\ell_2(x_1,x_2;\alpha_1,\alpha_2) = \begin{cases} x_1 + x_1^{1-\alpha_2}x_2^{\alpha_2}\vartheta(\alpha_1,\alpha_2), & (1-1/\alpha_1)x_1 > (1-1/\alpha_2)x_2, \\ x_2 + x_1^{\alpha_1}x_2^{1-\alpha_1}\vartheta(\alpha_2,\alpha_1), & \text{otherwise}, \end{cases}$$

$$\frac{\partial\ell_2(x_1,x_2;\alpha_1,\alpha_2)}{\partial x_1} = \begin{cases} 1 + (1-\alpha_2)x_1^{-\alpha_2}x_2^{\alpha_2}\vartheta(\alpha_1,\alpha_2), & (1-1/\alpha_1)x_1 > (1-1/\alpha_2)x_2 \\ \alpha_1 x_1^{\alpha_1-1}x_2^{1-\alpha_1}\vartheta(\alpha_2,\alpha_1), & \text{otherwise}, \end{cases}$$

$$\frac{\partial\ell_2(x_1,x_2;\alpha_1,\alpha_2)}{\partial x_2} = \begin{cases} \alpha_2 x_1^{1-\alpha_2}x_2^{\alpha_2-1}\vartheta(\alpha_1,\alpha_2), & (1-1/\alpha_1)x_1 > (1-1/\alpha_2)x_2, \\ 1 + (1-\alpha_1)x_1^{\alpha_1}x_2^{-\alpha_1}\vartheta(\alpha_2,\alpha_1), & \text{otherwise}, \end{cases}$$

$$\frac{\partial^2\ell_2(x_1,x_2;\alpha_1,\alpha_2)}{\partial x_1\partial x_2} = \begin{cases} \alpha_2(1-\alpha_2)x_1^{-\alpha_2}x_2^{\alpha_2-1}\vartheta(\alpha_1,\alpha_2), & (1-1/\alpha_1)x_1 > (1-1/\alpha_2)x_2, \\ \alpha_1(1-\alpha_1)x_1^{\alpha_1-1}x_2^{-\alpha_1}\vartheta(\alpha_2,\alpha_1), & \text{otherwise}. \end{cases}$$

These formulas can be used to compute the pairwise log-likelihood $l_d^2(\mathbf{u};\boldsymbol{\alpha})$. It is easy to confirm that $\partial^2\ell_2(x_1,x_2;\alpha_1,\alpha_2)/\partial x_1\partial x_2$ is a continuous function of α_1 and α_2, and hence, the log-likelihood is also a continuous function with no singular components. Standard optimization methods, such as the Newton–Raphson algorithm, can therefore be used to obtain the parameter estimates for the copula $C_d^{\mathbf{W}}$.

Similarly, the bivariate copula density can be obtained from the result of Proposition 6 for the exponential factor model with $p = 2$ factors. This density is also a continuous function, and therefore, the pairwise likelihood approach can be used to estimate the copula parameters.

4.3 | Parameter estimation for $C_d^{\mathbf{W}}$ with $F_\mathcal{E}$ being the Pareto distribution

The limiting extreme-value copula in this case is an extension of the Marshall–Olkin copula (5). Because this copula has singular components, the parameters of this copula cannot be estimated using the maximum likelihood approach. Instead, we use the method of moments approach with the result of Proposition 8. For a given data set, we first compute a $d \times d$ matrix of Spearman's correlations, Σ_ρ. Then we select the number of factors, p, and minimize the objective function

$$S(\boldsymbol{\theta}) = \sum_{j_1>j_2} \{(\Sigma_\rho)_{j_1,j_2} - \rho_{j_1,j_2}(\boldsymbol{\theta}_{j_1},\boldsymbol{\theta}_{j_2})\}^2,$$

where $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^\top, \ldots, \boldsymbol{\theta}_d^\top)^\top$, $\boldsymbol{\theta}_j = (\theta_{j1}, \ldots, \theta_{jp})^\top$, $j = 1, \ldots, d$, and the formula for ρ_{j_1,j_2} is given in Proposition 8. The number of factors can be determined from the bivariate scatter plots of the original data; each bivariate margin in this model with p factors has p singular components.

5 | EMPIRICAL STUDY

In this section, we assess the performance of the proposed methods for estimating the copula parameters of some simulated data sets. We then apply the limiting extreme-value copula from Proposition 2 to a financial data set. For computing the copula estimates, we use the `nlm()` function in the R statistical software (R Core Team, 2017).

5.1 | Algorithm performance for simulated data sets: Exponential factor model

We define $\boldsymbol{\alpha}_s = (\alpha_{1,s}, \alpha_{2,s}, \ldots, \alpha_{d,s})^\top$, $s = 1, \ldots, p$. We fit the extreme-value copula from Proposition 2 with $p = 1$ exponential factor to data sets generated from Models M1, M2, and M3, defined below, with the copula $C_d^{\mathbf{W}}$, which corresponds to the p-factor model (4), for $p = 1, 2$, and 3 exponential common factors, respectively:

TABLE 1 The mean (left) and standard deviation (right) of the estimated parameters of the limiting extreme-value copula in model (4) with $p = 1$ exponential factors (the original data are from Models 1, 2, and 3). Sample size is $N = 250$ (top half) and $N = 1,000$ (bottom half)

Model	Mean copula estimates	Standard deviation
	$N = 250$	
M1	1.80 1.80 1.79 1.41 1.41 1.41 1.41 2.36 2.37 2.37	0.11 0.11 0.11 0.06 0.07 0.07 0.07 0.17 0.19 0.19
M2	2.00 2.00 2.00 1.79 1.79 1.79 1.79 2.24 2.24 2.24	0.14 0.14 0.14 0.10 0.10 0.10 0.10 0.15 0.15 0.15
M3	1.95 2.15 2.14 1.47 1.46 1.46 1.47 2.27 2.33 1.72	0.13 0.14 0.15 0.07 0.06 0.07 0.07 0.17 0.17 0.10
	$N = 1,000$	
M1	1.80 1.80 1.80 1.40 1.40 1.41 1.41 2.39 2.39 2.39	0.06 0.06 0.06 0.03 0.03 0.03 0.03 0.10 0.09 0.09
M2	2.01 2.01 2.02 1.78 1.78 1.78 1.78 2.25 2.26 2.25	0.07 0.07 0.08 0.05 0.05 0.05 0.05 0.08 0.08 0.08
M3	1.95 2.15 2.14 1.45 1.45 1.46 1.45 2.30 2.35 1.71	0.06 0.07 0.07 0.03 0.03 0.03 0.03 0.08 0.08 0.04

M1. $d = 10$, $p = 1$ and $\boldsymbol{\alpha}_1 = (1.8, 1.8, 1.8, 1.4, 1.4, 1.4, 1.4, 2.4, 2.4, 2.4)^\top$;

M2. $d = 10$, $p = 2$ and $\boldsymbol{\alpha}_1 = (2.2, 2.2, 2.2, 1.9, 1.9, 1.9, 1.9, 2.5, 2.5, 2.5)^\top$, $\boldsymbol{\alpha}_2 = (1.4, 1.4, 1.4, 1.6, 1.6, 1.6, 1.6, 2.0, 2.0, 2.0)^\top$;

M3. $d = 10$, $p = 3$ and $\boldsymbol{\alpha}_1 = (2.4, 2.4, 2.4, 1.8, 1.8, 1.8, 1.8, 2.8, 2.8, 2.8)^\top$, $\boldsymbol{\alpha}_2 = (1.3, 1.4, 1.5, 1.6, 1.6, 1.6, 1.6, 1.7, 1.8, 1.9)^\top$ and $\boldsymbol{\alpha}_3 = (2.0, 1.8, 1.6, 1.4, 1.4, 1.4, 1.4, 1.8, 2.2, 2.6)^\top$.

For each of the three models, we simulate 1,000 samples of size $N = 250$ and $N = 1,000$. We then compute the parameter estimates from the data generated from M1, M2, and M3 using the pairwise likelihood approach explained in Section 4.2. Table 1 shows the mean and standard deviation of the corresponding estimates.

For Model 1, the estimates are very close to the true values, and the standard deviation is smaller for the larger sample size. The running time for $N = 1000$ is about 2 min on an Intel core i5-2410M CPU at 2.3 GHz. By trying different sets of parameters, we found that the corresponding copula estimates are close to the true parameters and that the algorithm is quite fast for $d \leq 20$.

To assess the goodness of fit for the misspecified model (4) with $p = 1$ used to fit the data generated from Models M2 and M3, we simulate $N = 10,000$ replicates from each of these two models. For each pair of variables in the data sets simulated from these models, we compute empirical estimates of the Spearman's rho, S_ρ, and the tail-weighted measures of dependence, ϱ_L and ϱ_U (Krupskii & Joe, 2015b). The tail-weighted dependence measures can be used to assess the strength of dependence between a pair of variables in the lower and upper tails. We also simulate $N = 10,000$ replicates to compute the model-based estimates of these quantities for the estimated (misspecified) model (4) with $p = 1$, $\boldsymbol{\alpha}_1 = (2.01, 2.01, 2.02, 1.78, 1.78, 1.78, 1.78, 2.25, 2.26, 2.25)^\top$ and $\boldsymbol{\alpha}_1 = (1.95, 2.15, 2.14, 1.45, 1.45, 1.46, 1.45, 2.30, 2.35, 1.71)^\top$ for Models M2 and M3, respectively (we use the mean copula estimates for M2 and M3 from Table 1). We then compute the mean (absolute) differences between the empirical estimates and the corresponding model-based estimates of the three measures, S_ρ, ϱ_L, and ϱ_U, denoted by Δ_ρ, Δ_L, and Δ_U ($|\Delta_\rho|, |\Delta_L|$, and $|\Delta_U|$), respectively. Table 2 shows the results.

We can see that the misspecified model (4) with $p = 1$ factor fits the data generated from model (4) with $p = 2$ and $p = 3$ factors very well, both in the middle of the distribution and in its tails. If $p = 2$, this can be explained by the fact that, in order to obtain the tail dependence for each pair of variables, one of the two factors should dominate the other one so that $\alpha_{j1} > \alpha_{j2}$ or $\alpha_{j1} < \alpha_{j2}$ for all $j = 1, \ldots, 10$; see Proposition 6. This means that the dominating factor alone can describe dependencies among the variables quite well.

TABLE 2 $\Delta_\rho, |\Delta_\rho|, \Delta_L, |\Delta_L|, \Delta_U, |\Delta_U|$ for the estimated limiting extreme-value copula in model (4) with $p = 1$ factor (the original data are simulated from model (4) with $p = 2$ [top] and $p = 3$ [bottom])

p	Δ_ρ	$\vert\Delta_\rho\vert$	Δ_L	$\vert\Delta_L\vert$	Δ_U	$\vert\Delta_U\vert$
2	0.01	0.04	0.02	0.06	−0.02	0.04
3	0.01	0.05	0.00	0.06	−0.02	0.06

Note. Monte Carlo simulations with $N = 10,000$ replicates were used to compute these values.

We obtained similar results using different parameters $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$, and $\boldsymbol{\alpha}_3$. This implies that model (4) with one exponential factor can provide a good approximation of the more general p-factor model with $p > 1$ factors even though models with $p > 1$ exponential factors can add more flexibility when modelling data with permutation asymmetry.

5.2 | Algorithm performance for simulated data sets: Pareto factor model

Using the notation from the previous section, we generate $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ with $\alpha_{js} \sim U(0.5, 2.5)$, $j = 1, \dots, 15$, $s = 1, 2$. We then simulate 1,000 samples of size $N = 250$ and $N = 1,000$ from copula (5) with $p = 2$ factors, where $F_{\mathcal{E}}(x) = 1 - x^{-4}$, $x > 1$ (the Pareto distribution with the shape parameter equal to 4), and the parameters

$$\boldsymbol{\theta}_1 = (0.07, 0.88, 0.70, 0.57, 0.04, 0.43, 0.41, 0.01, 0.36, 0.39, 0.73, 0.46, 0.02, 0.71, 0.28)^\top,$$
$$\boldsymbol{\theta}_2 = (0.66, 0.02, 0.09, 0.17, 0.54, 0.52, 0.04, 0.84, 0.05, 0.33, 0.03, 0.47, 0.87, 0.02, 0.24)^\top.$$

For each of the simulated samples, we estimate the copula parameters, $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, as explained in Section 4.3. Table 3 shows the bias and standard deviation of the estimates.

The bias is small, even with $N = 250$, and the standard deviation is smaller for the larger sample size $N = 1,000$, as expected. We found that the proposed estimation method works well for different sets of parameters and numbers of factors.

TABLE 3 The mean (left) and standard deviation (right) of the estimated parameters of the limiting extreme-value copula in model (4) with $p = 2$ Pareto factors

***N* (Parameter)**	**Bias**														
250 ($\boldsymbol{\theta}_1$)	0.04	−0.07	−0.05	−0.04	0.03	0.01	−0.02	0.05	−0.01	−0.01	−0.06	0.00	0.05	−0.06	0.00
250 ($\boldsymbol{\theta}_2$)	−0.05	0.05	0.04	0.02	−0.04	−0.03	0.02	−0.07	0.02	−0.01	0.04	−0.02	−0.06	0.04	0.00
1000 ($\boldsymbol{\theta}_1$)	0.00	−0.02	−0.02	−0.02	0.00	−0.01	0.00	0.00	0.00	−0.01	−0.02	−0.01	0.00	−0.02	0.00
1000 ($\boldsymbol{\theta}_2$)	−0.02	0.00	0.00	0.00	−0.01	−0.01	0.00	−0.02	0.00	−0.01	0.00	−0.01	−0.02	0.00	0.00
***N* (Parameter)**	**Standard deviation**														
250 ($\boldsymbol{\theta}_1$)	0.15	0.20	0.14	0.09	0.13	0.05	0.10	0.20	0.08	0.04	0.17	0.05	0.20	0.17	0.04
250 ($\boldsymbol{\theta}_2$)	0.15	0.20	0.13	0.08	0.13	0.06	0.08	0.20	0.07	0.04	0.16	0.05	0.20	0.15	0.04
1000 ($\boldsymbol{\theta}_1$)	0.02	0.03	0.03	0.03	0.01	0.03	0.03	0.01	0.02	0.03	0.03	0.03	0.01	0.03	0.02
1000 ($\boldsymbol{\theta}_2$)	0.03	0.02	0.02	0.02	0.03	0.03	0.02	0.03	0.02	0.03	0.02	0.03	0.02	0.01	0.02

Note. Sample size is $N = 250$ (top half) and $N = 1,000$ (bottom half).

5.3 | Application to financial data

In this section, we apply the limiting extreme-value copula corresponding to the one-factor model (1) with the exponential factors to financial data. We select nine stocks from the S&P 500 index with tickers `AIV, AVB, BXP, EQR, HCN, HCP, HST, PSA, SPG`. Because these stocks are all from the same financial sector (real estate), it is reasonable to assume that there exists some factor that could affect all of them simultaneously. We consider monthly log-returns' maxima for the years 2000–2006, and 2011–2016; 156 months in total. We exclude the years of the subprime mortgage crisis, 2007–2010, to remove nonstationarity from the data: all of the stocks on the market exhibit much stronger dependencies during this period. The remaining observations show no significant serial correlations or nonstationarity, and so they can be treated as replicates.

We fit the generalized extreme-value distribution for each of the nine marginals and use the probability integral transform to convert the data to uniform scores. We use normal scores to better visualize the data, which are obtained by applying the inverse normal cdf to the uniform scores. Figure 3 shows the normal score scatter plots for three pairs of variables.

We see a stronger dependence in the upper tail than in the lower tail, because the scatter plots have sharper upper tails. We fit the copula $C_d^{\mathbf{W}}$ from Proposition 3 to the uniform scores data (denoted as Model A1). For comparison, we also fit the Archimedean Gumbel copula (denoted as Model A2) and the one-factor Gumbel copula (Krupskii & Joe 2013; denoted as Model A3) to these data. To assess the goodness of fit of the estimated models, we compute Δ_ρ, Δ_L, and Δ_U and $|\Delta_\rho|$, $|\Delta_L|$, and $|\Delta_U|$, as shown in Section 4.2; Table 4 shows the results.

Model A1 fits the data very well and has a better fit in the middle of the distribution than A2, as indicated by the smaller values of Δ_ρ and $|\Delta_\rho|$. Also, A1 has a better fit in the lower tail than A3. Model A2 is an extreme-value copula with a stronger dependence in the upper tail, but it is also

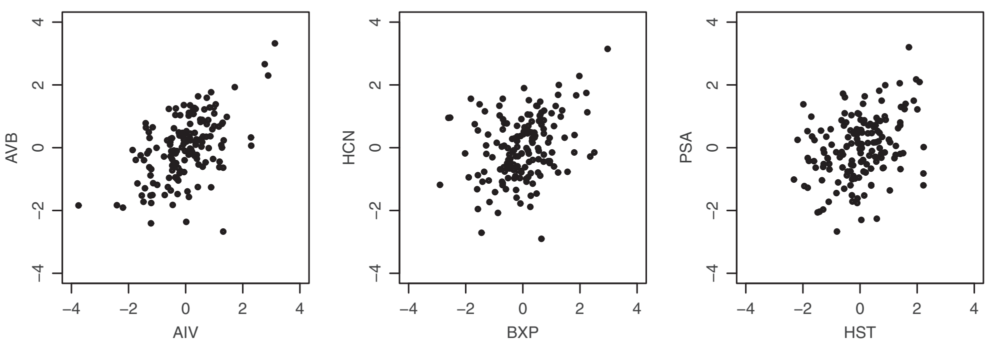

FIGURE 3 Normal scores scatter plots of log-returns' monthly maxima; pairs (left to right): (AIV, AVB), (BXP, HCN), and (HST, PSA)

TABLE 4 Δ_ρ, $|\Delta_\rho|$, Δ_L, $|\Delta_L|$, Δ_U, $|\Delta_U|$ for the estimated three models of the financial data

Model	Δ_ρ	$\|\Delta_\rho\|$	Δ_L	$\|\Delta_L\|$	Δ_U	$\|\Delta_U\|$
A1	−0.01	0.05	0.04	0.11	0.01	0.07
A2	−0.07	0.11	0.02	0.13	−0.03	0.08
A3	−0.01	0.04	0.08	0.13	0.01	0.07

Note. Monte Carlo simulations with $N = 10,000$ replicates were used to compute these values.

an exchangeable copula; so it assumes the same dependence structure for each pair of variables, which is too restrictive. Model A3 is a parsimonious model with a factor structure and Gumbel linking copulas, but the joint copula cdf in this model is not an extreme-value copula, meaning that this copula can underestimate probabilities of extreme events.

To illustrate these ideas, we define

$$p_{1|2:j}(q) = \Pr\{r_1^m > r_1^m(q) | r_2^m > r_2^m(q), \dots, r_j^m > r_j^m(q)\},$$

where r_j^m is a monthly maximum of the jth log-return, and $r_j^m(q)$ is the qth quantile of r_j^m, $j = 1, \dots, 9$. The conditional tail probabilities, $p_{1|2:j}(q)$, is an important risk measure widely used in financial applications. The predicted value of this measure is much smaller for A3 ($p_{1|2:5}(0.9) = 0.71$) than for A1 ($p_{1|2:5}(0.9) = 0.91$). On the other hand, the limiting extreme-value copula in model (1) with exponential factors inherits all of the appealing properties of factor copula models, including their tractability and interpretability.

6 | DISCUSSION

In this paper, we studied the tail properties of factor copula models with a linear structure. We derived limiting extreme-value copulas from these models and showed how parameter estimates can be obtained for these copulas. These extreme-value copulas can be used for modelling extremes data when there exists one or several factors that affect all of the variables simultaneously. We applied one of these extreme-value copulas to a financial data set, and the model showed a good fit to the data.

Possible extensions to the linear factor copula include hierarchical linear models and models with different distributions for different factors. The corresponding extreme-value copulas can handle extreme data with complex structures, and their properties are a topic for future research. Another research direction is the more general class of models in which the observed variables are nonlinear functions of latent (unobserved) common factors.

ACKNOWLEDGEMENTS

This research was supported by the King Abdullah University of Science and Technology (KAUST). We would like to thank the anonymous referee, associate editor, and editor-in-chief for their comments that helped improve this paper.

ORCID

Pavel Krupskii http://orcid.org/0000-0001-9658-735X
Marc G. Genton http://orcid.org/0000-0001-6467-2998

REFERENCES

Aas, K., Czado, C., Frigessi, A. & Bakken, H. (2009). Pair-copula constructions of multiple dependence. *Insurance: Mathematics and Economics*, *44*, 182–198.

Bárdossy, A. & Li, J. (2008). Geostatistical interpolation using copulas. *Water Resources Research*, *44*, W07412.

Cherubini, U., Durante, F. & Mulinacci, S. (2015). *Marshall Olkin distributions—Advances in theory and applications*, Springer proceedings in mathematics & statistics, Springer International Publishing Switzerland.

Chistyakov, V. P. (1964). A theorem on sums of independent, positive random variables and its applications to branching processes. *Theory of Probability and Its Applications*, *9*, 640–648.

Cox, D. R. & Reid, N. (2004). A note on pseudolikelihood constructed from marginal densities. *Biometrika*, *91*, 729–737.

Feller, W. (1970). *An introduction to probability theory and its applications, vol. 2.* USA: John Wiley & Sons.

Genest, C. & Favre, A. C. (2007). Everything you always wanted to know about copula modeling but were afraid to ask. *Journal of Hydrologic Engineering*, *12*, 347–368.

Gudendorf, G. & Segers, J. (2010). Extreme-value copulas. In P. Jaworski, F. Durante, W. Härdle & T. Rychlik (Eds.), *Copula theory and its applications* (pp. 127–145) Berlin, Heidelberg: Springer.

Hua, L. & Joe, H. (2017). Multivariate dependence modeling based on comonotonic factors. *Journal of Multivariate Analysis*, *155*, 317–333.

Joe, H. (2014). *Dependence modeling with copulas*, Boca Raton, FL: Chapman & Hall/CRC.

Krupskii, P., Huser, R. & Genton, M. G. (2018). Factor copula models for replicated spatial data. *Journal of the American Statistical Association*, *521*.

Krupskii, P. & Joe, H. (2013). Factor copula models for multivariate data. *Journal of Multivariate Analysis*, *120*, 85–101.

Krupskii, P. & Joe, H. (2015a). Structured factor copula models: Theory, inference and computation. *Journal of Multivariate Analysis*, *138*, 53–73.

Krupskii, P. & Joe, H. (2015b). Tail-weighted measures of dependence. *Journal of Applied Statistics*, *42*, 614–629.

Kurowicka, D. & Cooke, R. (2006). *Uncertainty analysis with high dimensional dependence modelling*, Chichester, England; Hoboken, NJ: Wiley.

Lindsay, B. (1998). Composite likelihood methods. *Contemporary Mathematics*, *80*, 220–239.

Marshall, W. A. & Olkin, I. (1967). A multivariate exponential distribution. *Journal of the American Statistical Association*, *62*, 30–44.

Mazo, G., Girard, S. & Forbes, F. (2016). A flexible and tractable class of one-factor copulas. *Statistics and Computing*, *26*, 965–979.

Nikoloulopoulos, A. K. & Joe, H. (2015). Factor copula models for item response data. *Psychometrika*, *80*, 121–150.

Oh, D. H. & Patton, A. (2017). Modeling dependence in high dimensions with factor copulas. *Journal of Business & Economic Statistics*, *35*, 139–154.

Patton, A. (2006). Modeling asymmetric exchange rate dependence. *International Economic Review*, *47*, 527–556.

R Core Team (2017). *R: A language and environment for statistical computing*, R Foundation for Statistical Computing, Vienna, Austria.

Rogge, E. & Schönbucher, P. J. (2003). Modeling dynamic portfolio credit risk. NCCR FINRISK Technical report No. 112. http://www.nccr-finrisk.uzh.ch/media/pdf/wp/WP112_5.pdf

Segers, J. (2012). Max-stable models for multivariate extremes. *REVSTAT*, *10*, 61–82.

Sklar, A. (1959). Fonctions de répartition à *n* dimensions et leurs marges. *Institute of Statistics of the University of Paris*, *8*, 229–231.

How to cite this article: Krupskii P, Genton MG. Linear factor copula models and their properties. *Scand J Statist*. 2018;45:861–878. https://doi.org/10.1111/sjos.12325

APPENDIX

A.1 Proof of Proposition 1

Without loss of generality, we consider the pair $(W_1, W_2)^\top$. We designate $c_0(q) = \bar{F}_{\mathcal{E}}^{-1}(q)$ and $c_j(q) = (\bar{F}_1^{W_j})^{-1}(q), j = 1, 2$. For $\delta > 1$, we have $c_j(q) > \delta\alpha_j c_0(q)$. For any $\epsilon > 0$ such that $\delta - \epsilon > 1$,

$$
\begin{aligned}
\Pr\{W_1 > c_1(q), W_2 > c_2(q)\} &\le \Pr\{W_1 > \delta\alpha_1 c_0(q), W_2 > c_2(q)\} \\
&\le \Pr\{\mathcal{E}_0 > (\delta - \epsilon)c_0(q)\} + \Pr\{\mathcal{E}_1 > \epsilon\alpha_1 c_0(q), W_2 > c_2(q)\} \\
&= \bar{F}_{\mathcal{E}}\{(\delta - \epsilon)c_0(q)\} + q\bar{F}_{\mathcal{E}}\{\alpha_1 c_0(q)\} = o(q).
\end{aligned}
$$

We now show that conditions of this assumption are satisfied for the Weibull distribution $F_{\mathcal{E}}(x) = 1 - \exp(-x^{\gamma})$ with $\gamma > 1$. When $w = \delta \bar{F}_{\mathcal{E}}^{-1}(q) = \delta(-\ln q)^{1/\gamma}$, we have

$$\Pr\{W_0 := \alpha\mathcal{E}_0 + \mathcal{E}_1 > w\} = \gamma\delta(-\ln q)\int_0^{1/\alpha} y^{\gamma-1}q^{\delta^{\gamma}\{(1-\alpha y)^{\gamma}+y^{\gamma}\}}\mathrm{d}y.$$

We let $f_{\alpha}(y;\gamma) = (1-\alpha y)^{\gamma} + y^{\gamma}$. If $\gamma > 1$, then $\min_{y\in(0,1/\alpha)} f_{\alpha}(y;\gamma) = \alpha^{*} = \{1+\alpha^{\gamma/(\gamma-1)}\}^{1-\gamma} < 1$, and the derivative $\partial f_{\alpha}(y;\gamma)/\partial y$ is bounded for $0 \le y \le 1/\alpha$. If $\delta^{\gamma}\alpha^{*} < 1$, then

$$\Pr\{W_0 > w\} = \gamma\delta(-\ln q)q^{\delta^{\gamma}\alpha *}\int_0^{1/\alpha} y^{\gamma-1}q^{\delta^{\gamma}\{f_{\alpha}(y;\gamma)-\alpha^{*}\}}\mathrm{d}y \sim q^{\delta^{\gamma}\alpha^{*}}$$

and therefore $\Pr\{W_0 > w\}/q \to \infty$ as $q \to 0$. This implies that $(\bar{F}_1^{W_0})^{-1}(q) \ge \delta c_0(q)$ for $\delta > 1$ such that $\delta^{\gamma}\alpha^{*} < 1$. □

A.2 Proof of Proposition 3

We assume that $\alpha_j > 1, j = 1, \dots, d$. When $c_j = \alpha_j(\ln n - \ln y_j), j = 1, \dots, d$, then $F_1^{W_j}(c_j) = 1 - x_j/n + o(1/n)$. We have

$$\begin{aligned} F_d^{\mathbf{W}}(c_1, \dots, c_d) &= \int_0^{\ln n - \ln y_{(1)}} \prod_{j=1}^{d} [1 - \exp\{\alpha_j(w_0 - \ln n + \ln y_j)\}]\exp(-w_0)\mathrm{d}w_0 \\ &= \frac{1}{n}\int_{-\ln n}^{-\ln y_{(1)}} \prod_{j=1}^{d}[1-\exp\{\alpha_j(w_0+\ln y_j)\}]\exp(-w_0)\mathrm{d}w_0 \\ &= 1 - \frac{y_{(1)}}{n} + \frac{1}{n}\sum_{m=1}^{d}(-1)^m \sum_{1\le j_1<\dots<j_m\le d} h_{j_1,\dots,j_m}(y_{j_1}, \dots, y_{j_m}, y_{(1)}), \end{aligned}$$

where

$$\begin{aligned} h_{j_1,\dots,j_m}(y_{j_1}, \dots, y_{j_m}, y_{(1)}) &= \int_{-\ln n}^{-\ln y_{(1)}} \exp\left\{\sum_{l=1}^{m}\alpha_{j_l}(w_0+\ln y_{j_l})\right\}\exp(-w_0)\mathrm{d}w_0 \\ &= \frac{y_{(1)}}{\sum_{l=1}^{m}\alpha_{j_l} - 1}\prod_{l=1}^{m}\left\{\frac{y_{j_l}}{y_{(1)}}\right\}^{\alpha_{j_l}} + o(1). \end{aligned}$$

Therefore,

$$\begin{aligned} \ell_d(x_1, \dots, x_d) &= \lim_{n\to\infty} n\left\{1 - F_d^{\mathbf{W}}(c_1, \dots, c_d)\right\} \\ &= y_{(1)}\left[1 - \sum_{m=1}^{d}(-1)^m \sum_{1\le j_1<\dots<j_m\le d}\frac{1}{\sum_{l=1}^{m}\alpha_{j_l} - 1}\prod_{l=1}^{m}\left\{\frac{y_{j_l}}{y_{(1)}}\right\}^{\alpha_j}\right]. \end{aligned}$$

Now, we assume without loss of generality that $\alpha_1 > 1$ and $\alpha_2 < 1$ (other cases are considered analogously). If $c_1 := \alpha_1(\ln n - \ln y_1)$ and $c_2 := \ln n - \ln(1-\alpha_2) - \ln x_2$, then $F_1^{W_j}(c_j) = 1 - x_j/n + o(1/n), j = 1, 2$. This implies that $c_2/\alpha_2 > c_1/\alpha_1$ as $n \to \infty$, and

$$\begin{aligned}\Pr(W_1 < c_1, W_2 < c_2) &= \int_0^{c_1/\alpha_1} [1 - \exp\{\alpha_1(w_0 - \ln n + \ln y_1)\}]\\ &\quad \times [1 - \exp\{\alpha_2 w_0 - \ln n + \ln(1-\alpha_2) + \ln x_2\}]\exp(-w_0)\mathrm{d}w_0\\ &= 1 - \frac{x_1}{n} - \frac{x_2}{n} + o\left(\frac{1}{n}\right).\end{aligned}$$

Therefore, $\lambda_U^{1,2} = 0$. □

A.3 Proof of Proposition 5

Without loss of generality, we let $j = 1$ and $k = 2$. We assume that for some $1 \le s_0 \le p$, we have $\alpha_{js_0} = \max_s \alpha_{js} > 1, j = 1, 2$. Let $c_j = \alpha_{js_0}\bar{F}_{\mathcal{E}}^{-1}(1/n)$, then $F_1^{W_j}(c_j) = 1 - 1/n + o(1/n)$. This implies that

$$\begin{aligned}&\Pr(W_1 > c_1,\ W_2 > c_2) \le \Pr(W_1 > c_1) = 1/n + o(1/n),\\ &\Pr(W_1 > c_1,\ W_2 > c_2) \ge \Pr\{\mathcal{E}_{s_0} > \bar{F}_{\mathcal{E}}^{-1}(1/n)\} = 1/n + o(1/n),\end{aligned}$$

so that $\lambda_U = 1$.

Now, we assume that $\alpha_{1s_1} = \max_s \alpha_{1s} > 1$ and $\alpha_{2s_2} = \max_s \alpha_{2s} > 1$, where $s_1 \ne s_2$ (other cases are considered analogously). If $c_j = \alpha_{js_j}\bar{F}_{\mathcal{E}}^{-1}(1/n)$, then $F_1^{W_j}(c_j) = 1 - 1/n + o(1/n), j = 1, 2$, and

$$\begin{aligned}\Pr(W_1 > c_1,\ W_2 > c_2) &\le \Pr(W_1 + W_2 > c_1 + c_2)\\ &= \sum_{s=1}^{p} \Pr\{(\alpha_{1s} + \alpha_{2s})\mathcal{E}_{0s} + \mathcal{E}_1 + \mathcal{E}_2 > (\alpha_{1s_1} + \alpha_{2s_2})\bar{F}_{\mathcal{E}}^{-1}(1/n)\}\\ &= o(1/n),\end{aligned}$$

because $\max\{\max_s(\alpha_{1s} + \alpha_{2s}), 1\} < \alpha_{1s_1} + \alpha_{2s_2}$. It follows that $\lambda_U^{1,2} = 0$.

□

A.4 Proof of Proposition 6

We have

$$F_1^{W_j}(w_j) = 1 + \frac{\alpha_{j2}^2 \exp(-w_j/\alpha_{j2})}{(\alpha_{j2}-1)(\alpha_{j1}-\alpha_{j2})} - \frac{\alpha_{j1}^2 \exp(-w_j/\alpha_{j1})}{(\alpha_{j1}-1)(\alpha_{j1}-\alpha_{j2})} + \frac{\exp(-w_j)}{(\alpha_{j1}-1)(\alpha_{j2}-1)}.$$

We assume that $\alpha_{j1} > \max(1, \alpha_{j2})$, and $c_j = \alpha_{j1}(\ln n - \ln y_j), j = 1, 2$. It follows that $F_1^{W_j}(c_j) = 1 - x_j/n + o(1/n)$. We define $Z_j = \alpha_{j1}\mathcal{E}_{01} + \mathcal{E}_1$, $\alpha^* = \alpha_{11} + \alpha_{21} - 1$ and $w^* = \min(c_1/\alpha_{12}, c_2/\alpha_{22})$. Thus, we find that

$$F_2^{\mathbf{W}}(c_1, c_2) = \int_0^{w^*} \Pr(Z_1 < c_1 - \alpha_{12}w, Z_2 < c_2 - \alpha_{22}w)\exp(-w)\mathrm{d}w,$$

where $\Pr(Z_1 < z_1, Z_2 < z_2) = g(z_1, z_2) - g^*(z_1, z_2)$ and

$$g(z_1, z_2) = 1 + \frac{\exp(-z_1)}{\alpha_{11}-1} + \frac{\exp(-z_2)}{\alpha_{21}-1} - \frac{\exp(-z_1-z_2)}{\alpha^*},$$

$$g^*(z_1, z_2) = \begin{cases} g_1^*(z_1, z_2) := \frac{\alpha_{11}\exp(-z_1/\alpha_{11})}{\alpha_{11}-1} + \frac{\alpha_{11}\exp\{-z_2+z_1(\alpha_{21}-1)/\alpha_{11}\}}{\alpha^*(\alpha_{21}-1)}, & z_1/\alpha_{11} < z_2/\alpha_{21},\\ g_2^*(z_1, z_2) := \frac{\alpha_{21}\exp(-z_2/\alpha_{21})}{\alpha_{21}-1} + \frac{\alpha_{21}\exp\{-z_1+z_2(\alpha_{11}-1)/\alpha_{21}\}}{\alpha^*(\alpha_{11}-1)}, & z_1/\alpha_{11} \ge z_2/\alpha_{21}.\end{cases}$$

We see that $\int_0^{w^*} g(c_1-\alpha_{12}w, c_2-\alpha_{22}w)\exp(-w)\mathrm{d}w = 1+o(1/n)$. We assume that $\psi = \alpha_{22}/\alpha_{21} - \alpha_{12}/\alpha_{11} > 0$ (the other case is considered analogously). It follows that $w^* = (\alpha_{21}/\alpha_{22})(\ln n - \ln y_2)$ as $n \to \infty$ and if $y_1 > y_2$, then

$$
\begin{aligned}
F_2^{\mathbf{W}}(c_1,c_2) &= 1 - \int_0^{\frac{\ln(y_1/y_2)}{\psi}} g_1^*(c_1-\alpha_{12}w, c_2-\alpha_{22}w)\exp(-w)\mathrm{d}w \\
&\quad - \int_{\frac{\ln(y_1/y_2)}{\psi}}^{w^*} g_2^*(c_1-\alpha_{12}w, c_2-\alpha_{22}w)\exp(-w)\mathrm{d}w + o\left(\frac{1}{n}\right) \\
&= 1 - \frac{x_1}{n} - \frac{k_1}{n} y_1^{1-\alpha_{21}} y_2^{\alpha_{21}} - \frac{1}{n}\varphi(y_1,y_2) + o\left(\frac{1}{n}\right).
\end{aligned}
$$

If $y_1 \le y_2$, then

$$
\begin{aligned}
F_2^{\mathbf{W}}(c_1,c_2) &= 1 - \int_0^{w^*} g_2^*(c_1-\alpha_{12}w, c_2-\alpha_{22}w)\exp(-w)\mathrm{d}w + o\left(\frac{1}{n}\right) \\
&= 1 - \frac{x_2}{n} - \frac{k_2}{n} y_1^{\alpha_{11}} y_2^{1-\alpha_{11}} + o\left(\frac{1}{n}\right).
\end{aligned}
$$

We assume now that $\alpha_{11} > \max(\alpha_{12}, 1)$, but $1 < \alpha_{21} < \alpha_{22}$. Then $c_1 = \alpha_{11}(\ln n - \ln y_1)$ and $c_2 = \alpha_{22}(\ln n - \ln y_2^*)$, where $y_2^* = x_2(1-1/\alpha_{22})(1-\alpha_{21}/\alpha_{22})$. This implies that $w^* = \ln n - \ln y_2^*$, as $n \to \infty$, and

$$
\begin{aligned}
F_2^{\mathbf{W}}(c_1,c_2) = &\int_0^{w^*} \left\{1 + \frac{\exp(-c_2+\alpha_{22}w)}{\alpha_{21}-1}\right\}\exp(-w)\mathrm{d}w \\
&- \int_0^{y^*} g_1^*(c_1-\alpha_{12}w, c_2-\alpha_{22}w)\exp(-w)\mathrm{d}w \\
&- \int_{y^*}^{w^*} g_2^*(c_1-\alpha_{12}w, c_2-\alpha_{22}w)\exp(-w)\mathrm{d}w + o\left(\frac{1}{n}\right) \\
= 1 - \frac{y_2^*}{n} &+ \frac{y_2^*}{n(\alpha_{21}-1)(\alpha_{22}-1)} - \frac{y_1}{nm_1} - \frac{y_2^*}{nm_2} + o\left(\frac{1}{n}\right) = 1 - \frac{x_1+x_2}{n} + o\left(\frac{1}{n}\right),
\end{aligned}
$$

where

$$
y^* = \frac{\alpha_{22}/\alpha_{21} - 1}{\psi}\ln n + \frac{\ln y_1 - (\alpha_{22}/\alpha_{21})\ln y_2^*}{\psi}.
$$

Other cases are considered analogously. The result of this proposition can be obtained by taking the limit $\ell_2(x_1,x_2) = \lim_{n\to\infty} n\{1 - F_2^{\mathbf{W}}(c_1,c_2)\}$.

□

A.5 Proof of Proposition 7

As $c_j \to \infty$, we use the asymptotic property of the sum of independent Pareto variables with regularly varying tails (Feller, 1970) to obtain $F_1^{W_j}(c_j) = 1 - x_j/n + o(1/n)$, $j = 1, \ldots, d$, with $c_j := (n/x_j)^{1/\beta} \cdot \{\sum_{s=1}^{p} (\alpha_{js})^{\beta} + 1\}^{1/\beta}$. We have

$$F_d^{\mathbf{W}}(c_1, \dots, c_d) = 1 + \sum_{m=1}^{d} (-1)^m \sum_{1 \le j_1 < \dots < j_m \le d} \Pr(W_{j_1} > c_{j_1}, \dots, W_{j_m} > c_{j_m})$$
$$= 1 - \sum_{j=1}^{d} \frac{x_j}{n} + \sum_{m=2}^{d} (-1)^m \sum_{1 \le j_1 < \dots < j_m \le d} \Pr(W_{j_1} > c_{j_1}, \dots, W_{j_m} > c_{j_m}).$$

Let $W_j^* = W_j - \mathcal{E}_j = \sum_{s=1}^{p} \alpha_{js} \mathcal{E}_{0s}$. For $m \ge 2$ and $1 \le j_1 < \dots < j_m \le d$,

$$\Pr(W_{j_1} > c_{j_1}, \dots, W_{j_m} > c_{j_m}) \ge \Pr(W_{j_1}^* > c_{j_1}, \dots, W_{j_m}^* > c_{j_m}),$$
$$\begin{aligned}\Pr(W_{j_1} > c_{j_1}, \dots, W_{j_m} > c_{j_m}) &= \Pr(W_{j_1} > c_{j_1}, \dots, W_{j_m} > c_{j_m},\\ &\qquad \mathcal{E}_{j_1} \le n^{0.5/\beta}, \dots \mathcal{E}_{j_m} \le n^{0.5/\beta}) + o(1/n)\\ &\le \Pr(W_{j_1}^* > c_{j_1} - n^{0.5/\beta}, \dots, W_{j_m}^* > c_{j_m} - n^{0.5/\beta}) + o(1/n)\\ &= \Pr(W_{j_1}^* > c_{j_1}, \dots, W_{j_m}^* > c_{j_m}) + o(1/n).\end{aligned}$$

Also, $\Pr(W_j^* \ge c_j) = (x_j/n) \sum_{s=1}^{p} \theta_{js}, \quad j = 1, \dots, d$. Therefore,

$$F_d^{\mathbf{W}}(c_1, \dots, c_d) = \Pr(W_1^* \le c_1, \dots, W_d^* \le c_d) - \sum_{j=1}^{d} \frac{x_j}{n} \left(1 - \sum_{s=1}^{p} \theta_{js}\right) + o(1/n).$$

One can see that

$$\Pr(W_j^* \le c_j) = \Pr\left(\mathcal{E}_{01} \le \frac{c_j}{\alpha_{j1}}, \dots, \mathcal{E}_{0p} \le \frac{c_j}{\alpha_{jp}}\right) + o(1/n) = 1 - \sum_{s=1}^{p} \frac{x_j}{n} \theta_{js} + o(1/n),$$

and, therefore,

$$\begin{aligned}\Pr(W_1^* \le c_1, \dots, W_d^* \le c_d) &= \Pr\left(\mathcal{E}_{01} \le \min_j \frac{c_j}{\alpha_{j1}}, \dots, \mathcal{E}_{0p} \le \min_j \frac{c_j}{\alpha_{jp}}\right) + o(1/n)\\ &= 1 - \sum_{s=1}^{p} \frac{\max_j(\theta_{js} x_j)}{n} + o(1/n).\end{aligned}$$

This implies that $\ell_d(x_1, \dots, x_d) = \sum_{j=1}^{d} x_j \left(1 - \sum_{s=1}^{p} \theta_{js}\right) + \sum_{s=1}^{p} \max_j(\theta_{js} x_j)$ and $C_d^{\mathbf{W}}(u_1, \dots, u_d) = \prod_{j=1}^{d} u_j^{1-\sum_{s=1}^{p} \theta_{js}} \prod_{s=1}^{p} \left\{\min_j \left(u_j^{\theta_{js}}\right)\right\}$. □

A.6 Proof of Proposition 8

We have $\mathrm{cor}(U_1, U_2) = 12\rho_{12} - 3$, where

$$\begin{aligned}\rho_{12} &= \int_0^1 \int_0^1 C_2^{\mathbf{W}}(u_1, u_2) \mathrm{d}u_1 \mathrm{d}u_2 = \sum_{s=0}^{p} \int_0^1 \int_{u_2^{\zeta_s}}^{u_2^{\zeta_{s+1}}} u_1^{1-\sum_{i=1}^{s} \theta_{1i}} u_2^{1-\sum_{i=s+1}^{p} \theta_{2i}} \mathrm{d}u_1 \mathrm{d}u_2\\ &= \sum_{s=0}^{p} \int_0^1 \frac{u_2^{1-\sum_{i=s+1}^{p} \theta_{2i}}}{\Theta_{1s}} (u_2^{\zeta_{s+1}\Theta_{1s}} - u_2^{\zeta_s \Theta_{1s}}) \mathrm{d}u_2 = \frac{\zeta_s - \zeta_{s+1}}{(\zeta_{s+1}\Theta_{1s} + \Theta_{2s})(\zeta_s \Theta_{1s} + \Theta_{2s})}.\end{aligned}$$

□