JOURNAL OF COMPUTATIONAL AND GRAPHICAL STATISTICS
2025, VOL. 34, NO. 1, 72–83
https://doi.org/10.1080/10618600.2024.2374578

Functional Time Series Analysis and Visualization Based on Records

Israel Martínez-Hernández[a] and Marc G. Genton[b]

[a]School of Mathematical Sciences, Lancaster University, Lancaster, UK; [b]Statistics Program, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia

ABSTRACT

In many phenomena, data are collected on a large scale and at different frequencies. In this context, functional data analysis (FDA) has become an important statistical methodology for analyzing and modeling such data. The approach of FDA is to assume that data are continuous functions and that each continuous function is considered as a single observation. Thus, FDA deals with large-scale and complex data. However, visualization and exploratory data analysis, which are very important in practice, can be challenging due to the complexity of the continuous functions. Here we introduce a type of record concept for functional data, and we propose some nonparametric tools based on the record concept for functional data observed over time (functional time series). We study the properties of the trajectory of the number of record curves under different scenarios. Also, we propose a unit root test based on the number of records. The trajectory of the number of records over time and the unit root test can be used for visualization and exploratory data analysis. We illustrate the advantages of our proposal through a Monte Carlo simulation study. We also illustrate our method on two different datasets: Daily wind speed curves at Yanbu, Saudi Arabia and annual mortality rates in France. Overall, we can identify the type of functional time series being studied based on the number of record curves observed. Supplementary materials for this article are available online.

ARTICLE HISTORY
Received December 2022
Accepted June 2024

KEYWORDS
Exploratory data analysis; Functional depth; Functional time series visualization; Nonstationary functional time series; Record curves

1. Introduction

Due to modern technologies, data can now be collected on a large scale and in an automatic fashion for many phenomena, resulting in high-dimensional and high-frequency data that can be considered as continuous functions or surfaces (images). For example, in economy, finance, climatology, medicine, biology, and engineering, data can be collected with characteristics that vary along a continuum (time or space). Functional Data Analysis (FDA) deals with this type of data, where each continuous function can represent daily or monthly profiles, and these profiles are considered as a single point observation (see, e.g., Ramsay and Silverman 2005). Here, we assume that our data are a functional time series observation, that is, a sequence of curves observed over time.

An important part of data analysis is visualization and exploratory data analysis. This helps to decide whether to transform the data or what class of models to use. In general, visualization and exploratory data analysis provide data characteristics that are not apparent from statistical models. In the context of functional data, some of these tools are functional bagplots and functional highest density region plots (Hyndman and Shang 2010), the functional boxplot (Sun and Genton 2011), and magnitude-shape plots (Dai and Genton 2018). Although these tools are useful, they do not provide characteristics that vary over time.

Here, we propose some additional tools using the concept of records. The main advantage of the record concept is that it is invariant under monotonic transformations of the data. This allows us to cover a large class of functional time series data, including some nonlinear functional time series. In addition, records are useful to define and study extreme curves. With that motivation, we propose a new type of record concept for functional data, and then we obtain nonparametric tools based on this proposal.

The record theory has been studied extensively for a sequence $\{W_1, \dots, W_n\}$ of identically distributed univariate random variables for both independent and dependent data (Sparre Andersen 1954; Feller 1971; Ballerini and Resnick 1987; Lindgren and Rootzén 1987; Burridge and Guerre 1996; Ahsanullah and Nevzorov 2015). It studies the events that exceed (cross) all previous observations, that is, $W_n > \max\{W_1, \dots, W_{n-1}\}$. The two most studied quantities of records are the probability for a record at time n and the number of records observed up to time n. It is well known that the expected number of records for stationary time series grows at rate $\log n$ (Lindgren and Rootzén 1987). On the other hand, if the time series is a random walk process, the growth rate is $n^{1/2}$ (Sparre Andersen 1954; Feller 1971; Burridge and Guerre 1996). Moreover, if the time series has a linear trend component, then the number of records grows at rate n (Ballerini and Resnick 1987). Extensions of the study of records to multivariate data can be found in the literature (see Goldie and Resnick 1989, 1995; Gnedin 1998; Wergen, Majumdar, and Schehr 2012; Dombry and Zott 2018; Falk, Khorrami Chokami, and Padoan 2018).

A challenge in extending record definition to functional data space is that there is no natural way to define an order in this

CONTACT Israel Martínez-Hernández i.martinezhernandez@lancaster.ac.uk School of Mathematical Sciences, Lancaster University, Lancaster LA1 4YW, UK.
Supplementary materials for this article are available online. Please go to www.tandfonline.com/r/JCGS.

space. This can make the definition of minimum and maximum curves a tricky problem. Here, we propose extending the definition of records by using a level-crossing concept for functional data based on depth notions. A function will be defined as a new record if it is more "extreme" than the previous record curve, in other words, if the curve exceeds the previous record curve. Then, we study the behavior of the number of functional records under stationarity and under stochastic trend components. By visualizing the growth rates of the number of functional records, we can infer whether the functional time series is stationary or not, and we can infer the different types of trends. For a more formal test, we propose a unit root test based on the number of record curves.

To say that a function exceeds another function, we use the concept of depth. Depth has been used to evaluate the centrality (or extremality) of a curve. Several notions of depth (called functional depth) have been proposed for functional data, including integrated depth (Fraiman and Muniz 2001), band depth and modified band depth (López-Pintado and Romo 2009), half-region depth based on hypographs and epigraphs (López-Pintado and Romo 2011), spatial depth (Chakraborty and Chaudhuri 2014) and extremal depth (Narisetty and Nair 2016). Other functional depth definitions can be found in Nieto-Reyes and Battey (2016), Gijbels and Nagy (2017), and Huang and Sun (2019). Depth has been used in different statistical problems, for example, detecting outliers, obtaining robust estimators, and defining functional boxplots (Rousseeuw and Hubert 1999; Fraiman and Muniz 2001; Sun and Genton 2011; Sguera, Galeano, and Lillo 2014; Martínez-Hernández, Genton, and González-Farías 2019). The order induced by the functional depth can be viewed as order statistics. Here, we use the functional depth concept to define exceedance or level-crossing. If a curve exceeds the previous function (based on the functional depth), then we say that the function crosses the previous function.

The nonparametric tools introduced in this article are applied to two applications of functional time series: wind speed curves in Saudi Arabia and mortality rates in France. Let $X_i(s)$ be the daily curves of wind speed at 80m $[m/s]$ where $i = 1, \dots, n$ represents the day, and $s \in [0, 24)$ represents hours within a day. The study of wind speed curves is important for renewable energy generation. By using record curves, we can describe the dynamics of the record daily wind speed. It is relevant to know when and how often a record curve is observed to predict the efficiency of wind turbines and to prevent disruption and possible damage to a wind farm. Moreover, with the information of record curves, we can classify the underlying functional process and then obtain a better predictor. Next, let $X_i(s)$ denote the mortality rate in year i, at age s. It is important to know (besides prediction) how these rates behave over the years, taking into account all ages. By studying the functional records, we analyze whether the new functional records over the years correspond to the natural randomness of the process or if there is an indication of a decreasing trend. In general, the number of functional records provides information about the stationarity or nonstationarity properties of the functional time series.

The main contributions presented in this article are (a) the establishment of a definition of upper and lower records for functional time series; (b) the study of the growth rate of the number of functional records over time, under stationarity or nonstationarity assumptions; and (c) the introduction of a unit root test for an integrated of order one functional process as an application of the functional record. The contributions (b) and (c) provide tools for visualization and exploratory data analysis. Overall, this article provides robust nonparametric tools based on records for functional time series.

The remainder of our article is organized as follows. In Section 2, we introduce mathematical concepts for functional data, functional time series, and functional depth. In Section 3, we provide the definition of records for functional data. In Section 4, we study the properties of the number of functional records, both for stationary and nonstationary functional time series. In Section 5, we propose a unit root test as an application of the study of functional records. In that section, we conduct a simulation study to evaluate the performance of the proposed test. In Section 6, we illustrate our proposal on two different datasets: the daily curves of wind speed at Yanbu, Saudi Arabia, and the annual mortality rates for males in France. Section 7 presents some discussions. Proofs and additional simulation results are provided in the supplementary materials.

2. Preliminaries

2.1. Functional Time Series

Throughout this article, we assume that our data are a collection of n functional observations $\{x_1(s), \dots, x_n(s)\}$ with $s \in \mathcal{T}$. Without loss of generality, we assume $\mathcal{T} = [0, 1]$. Let X_i be a functional random variable defined on a function space $\mathcal{F}$; $X_i : \Omega \to \mathcal{F}$, where Ω is the sample space and $\mathcal{F}$ the function space (e.g., $\mathcal{F} = L_2([0, 1])$ or $\mathcal{F} = \mathcal{C}([0, 1])$). We assume that $\{x_i\}$ is a realization of the functional random variables $\{X_i\}$. If X is a functional random variable with distribution P, we write $X \sim P$, and it is said to be symmetrically distributed (P is centrally symmetric) about $z \in \mathcal{F}$ if and only if $X-z = -(X-z)$ in distribution.

Let $\mu_i := \mathbb{E}(X_i)$ be the mean function, and let C_{X_{i-h},X_i} be the covariance operator at lag h. If μ_i and C_{X_{i-h},X_i} are well defined and exist, the sequence of functional random variables $\{X_i, -\infty < i < \infty\}$ is said to be stationary if (i) $\mathbb{E}(X_i) = \mu$ for all i and (ii) $C_{X_{i+h},X_{j+h}}(z) = C_{X_i,X_j}(z)$, $z \in \mathcal{F}$ for all i, j, and h. One of the most popular models for functional time series is the functional autoregressive model of order p, FAR(p) (Horváth, Hušková, and Kokoszka 2010; Kokoszka and Reimherr 2013; Aue, Norinho, and Hörmann 2015). For $p = 1$, we have that $X_i(s) = \Psi(X_{i-1})(s) + \varepsilon_i(s)$, where $\Psi : \mathcal{F} \to \mathcal{F}$ is a bounded operator, and $\{\varepsilon_i\}$ is a functional white noise. If the "norm" of Ψ is smaller than one, then the FAR(1) model is stationary. A more general definition is the functional linear process, $\{X_i, i \in \mathbb{Z}\}$ with innovations $\{\varepsilon_i\}$, defined as

$$X_i(s) = \sum_{j=0}^{\infty} \Psi_j(\varepsilon_{i-j})(s),\ s \in [0, 1].$$

Under some conditions on the sum of the norms of Ψ_j, $\{X_i\}$ converges in probability (see Bosq 2000). The FAR(p) processes can be seen as a particular case of a functional linear process. We refer to Ramsay and Silverman (2005) and Bosq (2000) for a deeper understanding of functional random variables.

2.2. Depth for Functional Data

Depth is used to evaluate the centrality or extremality of a curve with respect to a distribution function. Several notions of functional depth have been proposed. The modified band depth (MBD) is one of the most popular functional depths and has motivated the development of extensions, modifications, and generalizations of functional depth definitions. Let $x \in \mathcal{F}$ and let $\mathbf{x}_{1:n} = \{x_1, \ldots, x_n\}$ be a sample of $X \sim P$. The MBD of x with respect to the sample $\mathbf{x}_{1:n}$ computes the proportion of time that the curve x is in a band constructed by two curves from $\mathbf{x}_{1:n}$. Then, the depth value is obtained by averaging the proportion of time over all possible bands. That is,

$$\mathrm{MBD}(x; \mathbf{x}_{1:n}) = \binom{n}{2}^{-1} \sum_{1 \le i_1 < i_2 \le n} \lambda \left[\{ s \in [0,1] \mid \min(x_{i_1}(s), x_{i_2}(s)) \le x(s) \le \max(x_{i_1}(s), x_{i_2}(s)) \} \right], \quad (1)$$

where λ is the Lebesgue measure on $[0, 1]$. Then, $x_i \prec x$ (read as x is more extreme than x_i) if $\mathrm{MBD}(x_i; \mathbf{x}_{1:n}) > \mathrm{MBD}(x; \mathbf{x}_{1:n})$. The corresponding population version is denoted by $\mathrm{MBD}(x; P)$. The definition (1) is for a band obtained with two different curves. However, the band can be obtained by more than two curves (see López-Pintado and Romo 2009, for more details).

Another functional depth is the extremal depth (ED). The ED of $x \in \mathcal{F}$ with respect to $\mathbf{x}_{1:n}$ computes the pointwise extremeness of the curve x. Namely, let $D_x(s; \mathbf{x}_{1:n}) := 1 - |\sum_{i=1}^{n} [\mathbb{1}\{x_i(s) < x(s)\} - \mathbb{1}\{x_i(s) > x(s)\}]|/n$ be the pointwise depth of x, taking values in $\mathbb{D}_x \subset \{0, 1/n, \ldots, 1\}$. Let $\mathbb{D}$ be the union of $\mathbb{D}_x$ over all functions x. Let $G_x(r) = \int_0^1 \mathbb{1}\{D_x(s, \mathbf{x}_{1:n}) \le r\} ds$, for each $r \in \mathbb{D}$, be the corresponding cumulative distribution function. Now, consider two functions x and x_i and let $0 \le d_1 < d_2 < \cdots < d_M \le 1$ be the ordered elements of their combined depth levels, $\mathbb{D}_x$ and $\mathbb{D}_{x_i}$. Then, $x_i \prec x$ (again, read as x is more extreme than x_i[1]) if $G_x(d_1) > G_{x_i}(d_1)$. If $G_x(d_1) = G_{x_i}(d_1)$, then the comparison is based on d_2 and repeated until the tie is broken. If $G_x(d_j) = G_{x_i}(d_j)$, for all $j = 1, \ldots, M$, then the two functions are equivalent in terms of depth, denoted as $x \sim x_i$. Finally, the ED of x is defined as

$$\mathrm{ED}(x; \mathbf{x}_{1:n}) = 1 - \frac{\#\{i : x \preceq x_i\}}{n}, \quad (2)$$

where "$\preceq$" denotes either $x \prec x_i$ or $x \sim x_i$. The corresponding population version is denoted by $\mathrm{ED}(x; P)$. See Narisetty and Nair (2016) for more details.

In this article, we will use depths (1) and (2) to illustrate and define a level-crossing concept for functional data. Instead of using the notation MBD or ED, we will use $\mathrm{fD}(x, \mathbf{x}_{1:n})$ to refer to any of these two for the functional depth of x with respect to the sample $\mathbf{x}_{1:n}$, and $\mathrm{fD}(x, P)$ to denote the population version.

[1] notice that the notation in Narisetty and Nair (2016) is $x_i \succ x$ to say that x is more extreme than x_i. Here, we have changed it to have a consistent notation with the MBD.

3. Definition of Functional Records

In this section, we present the concept of record for functional data. We first describe what a record is in the case of univariate scalar time series.

3.1. Classical Records

Let $\{W_1, \ldots, W_n\}$ be a sequence of continuous random variables in $\mathbb{R}$ (observe that $W_i = W_j$ with probability zero for $i \neq j$). Loosely speaking, an upper record at time n means that the value of W_n exceeds all values of $\{W_1, \ldots W_{n-1}\}$. Records can be formally defined in different ways. One definition involves an order in $\mathbb{R}$. That is, let $W_{(1)}, \ldots, W_{(n)}$ be the corresponding order statistics for the n random variables, then W_n is an upper record if $W_n = W_{(n)}$ (and a lower record if $W_n = W_{(1)}$) with probability one (Ahsanullah and Nevzorov 2015). However, there is no natural way to define an order in a functional space, and this makes it challenging to extend this definition of record to functional data. Another definition is comparing W_n with the previous record. Let M_{n-1} and m_{n-1} represent the upper and lower record at time $n-1$, respectively. Then, W_n will be a new upper (lower) record if $M_{n-1} < W_n$ ($W_n < m_{n-1}$) with probability one. That is, the value of W_n exceeds or crosses the value of M_{n-1} (or m_{n-1}). Given the value of M_{n-1}, the latter definition only involves comparing two quantities. Here, we will use this idea for functional data. We will say that a function at time n is a new record if it exceeds the previous record function.

3.2. Functional Records

Suppose we observe a functional time series $\mathbf{x}_{1:n} = \{x_1, \ldots, x_n\}$, where $n \ge 3$. By definition, x_1 and x_2 are functional records (one upper and the other lower functional record). Let y^u_{n-1} and y^l_{n-1} represent the upper and lower functional records at time $n-1$, respectively. To define the functional record at time n, we compare x_n with the previous two records, y^u_{n-1} and y^l_{n-1}. That is, x_n is defined as a functional record at time n if x_n exceeds y^u_{n-1} or y^l_{n-1}. Next, we explain what exceeding a function means.

We say that x_n exceeds y^u_{n-1} or y^l_{n-1} if x_n is more extreme than y^u_{n-1} or y^l_{n-1} in terms of the depth notion. That is, we use the functional depth defined in Section 2.2 to define extremeness. Let $T^u_n := \|x_n - y^u_{n-1}\|_{\mathcal{F}}$ and $T^l_n := \|x_n - y^l_{n-1}\|_{\mathcal{F}}$, with $\|\cdot\|_{\mathcal{F}}$ a norm in $\mathcal{F}$. We say that x_n is a candidate for an upper record if $T^u_n < T^l_n$ and a candidate for a lower record if $T^u_n > T^l_n$. This allows us to know if x_n will exceed y^u_{n-1} or y^l_{n-1}. Now, we describe how to compare functions. Let $\mathrm{fD}(x_n; \{x_n, y^u_{n-1}, y^l_{n-1}\})$, $\mathrm{fD}(y^u_{n-1}; \{x_n, y^u_{n-1}, y^l_{n-1}\})$, and $\mathrm{fD}(y^l_{n-1}; \{x_n, y^u_{n-1}, y^l_{n-1}\})$ be the corresponding depths values of x_n, y^u_{n-1}, and y^l_{n-1}, respectively. As mentioned in Section 2.2, we say that $y^u_{n-1} \prec x_n$ if $\mathrm{fD}(x_n; \{x_n, y^u_{n-1}, y^l_{n-1}\}) < \mathrm{fD}(y^u_{n-1}; \{x_n, y^u_{n-1}, y^l_{n-1}\})$.

Definition 1. Let $\mathbf{x}_{1:n} = \{x_1, \ldots, x_n\}$ be an observed functional time series, $n \ge 3$. If $T^u_n < T^l_n$ and $\mathrm{fD}(x_n; \{x_n, y^u_{n-1}, y^l_{n-1}\}) < \mathrm{fD}(y^u_{n-1}; \{x_n, y^u_{n-1}, y^l_{n-1}\})$, then x_n is defined as an upper functional record at time n. If $T^u_n > T^l_n$ and $\mathrm{fD}(x_n; \{x_n, y^u_{n-1}, y^l_{n-1}\}) < \mathrm{fD}(y^l_{n-1}; \{x_n, y^u_{n-1}, y^l_{n-1}\})$, then x_n is defined as a lower functional record at time n. Finally, x_n is called

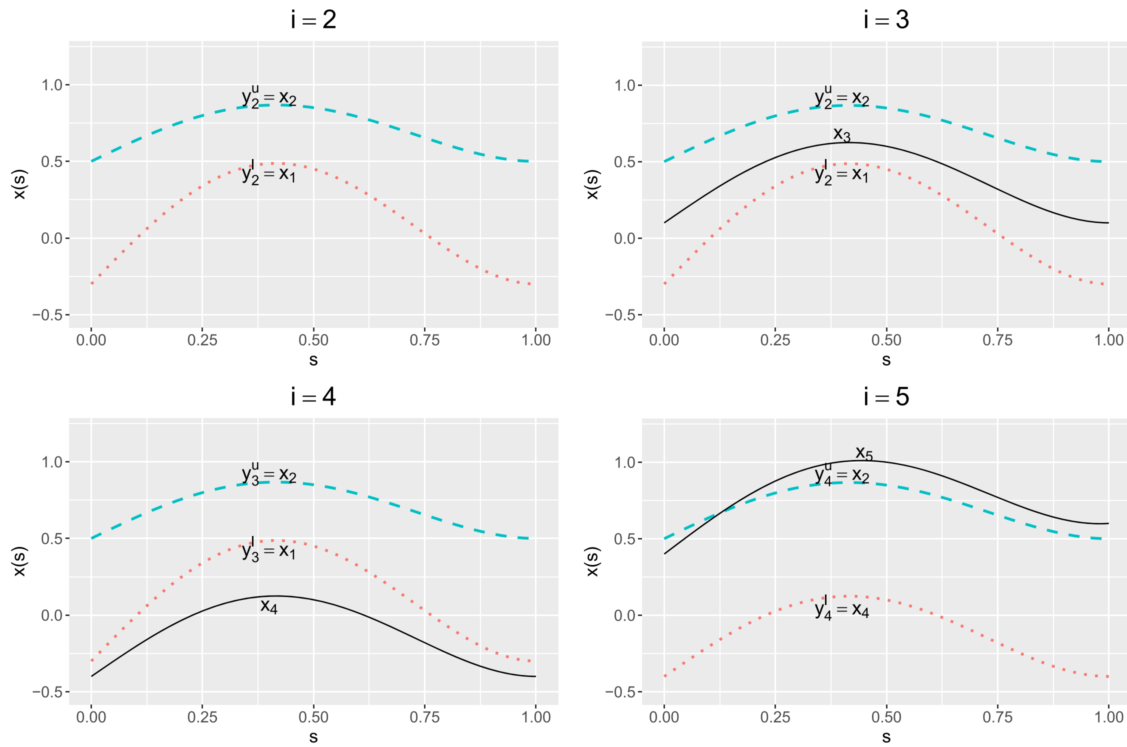

Figure 1. Functional records at $i = 2, 3, 4, 5$. In each figure, the previous upper functional record is indicated by the dashed curve, and the previous lower functional record is indicated by the dotted curve. At $i = 3$, x_3 does not exceed the previous record curves. At $i = 4$, x_4 exceeds the previous lower record curve. And at $i = 5$, x_5 exceeds the previous upper record curve.

a functional record if it is either a lower or upper functional record.

Notice that we could have $y_{n-1}^l \prec x_n$ or $y_{n-1}^u \prec x_n$, and $T_n^u = T_n^l$. To avoid this problem, we assume the following.

Assumption 1. For each $n \geq 3$, $P(T_n^u = T_n^l) = 0$.

Assumption 1 will always be satisfied in regular conditions of functional time series. Figure 1 shows a naive scenario to illustrate the functional record definition over time $i = 2, 3, 4, 5$ for a functional time series with sample size $n = 5$. At time $i = 2$, x_1 is defined as a lower record and x_2 as an upper record. At $i = 3$, x_3 is compared with the previous record functions y_2^u and y_2^l. In this case, x_3 does not exceed any of the previous record functions. At $i = 4$, x_4 exceeds the previous lower record y_3^l, so x_4 is defined as the new lower record at time $i = 4$. Finally, at time $i = 5$, x_5 exceeds the previous upper record y_4^u, and it is then defined as the new upper record. So, at time $i = 5$, x_5 is the last upper record, and x_4 is the last lower record.

4. Properties of Functional Record Number

Let $\mathbf{X} = \{X_i\}_{i=1}^n$ be a sequence of functional random variables. By definition, X_1 and X_2 are set to be functional records. Without loss of generality, let X_1 be a lower functional record and X_2 be an upper functional record. For $j = 1, \ldots, i-1$, let Y_j^u and Y_j^l be the sequence of functional random variables representing the upper and lower functional records, respectively, at time j, such that $Y_1^u = Y_2^u = X_2$ and $Y_1^l = Y_2^l = X_1$. Then, X_i is defined as an upper functional record at time i if $\mathrm{fD}(X_i; \{X_i, Y_{i-1}^u, Y_{i-1}^l\}) < \mathrm{fD}(Y_{i-1}^u; \{X_i, Y_{i-1}^u, Y_{i-1}^l\})$ given that X_i is a candidate for upper record. Similarly, X_i is defined as a lower functional record at time i if $\mathrm{fD}(X_i; \{X_i, Y_{i-1}^u, Y_{i-1}^l\}) < \mathrm{fD}(Y_{i-1}^l; \{X_i, Y_{i-1}^u, Y_{i-1}^l\})$, given that X_i is a candidate for lower record.

In this section, we study the number of functional records observed over time, $i = 3, 4, \ldots$. Let $R_i^u = \mathbb{1}\{X_i \text{ is an upper functional record}\}$ be the indicator of X_i being an upper functional record at time i, and let N_i^u be the counting process representing the number of *upper* functional records up to time i, that is,

$$N_i^u = \sum_{j=1}^{i} R_j^u. \tag{3}$$

We define the upper functional record times as $L^u(1) = 2$, and for $k = 2, 3, \ldots$, $L^u(k) = \min\{i : i > L^u(k-1) \text{ and } R_i^u = 1\}$. Notice that the definition of $L^u(k)$ is such that the events $\{N_i^u \geq k\}$ and $\{L^u(k) \leq i\}$ are equivalent. In the same way, we define the corresponding variables for the lower functional records. Let R_i^l, N_i^l, and $L^l(k)$ denote the respective variables for the lower functional records.

Notice that, from Definition 1, we have that a lower functional record is an upper functional record of the process $\{-X_i\}$. Therefore, we focus on the upper functional records. Also, with the definition of functional record, ties of depth values are allowed, but if X_i is an upper functional record at time i, then $\mathrm{fD}(X_i; \{X_i, Y_{i-1}^u, Y_{i-1}^l\})$ can only tie with $\mathrm{fD}(Y_{i-1}^l; \{X_i, Y_{i-1}^u, Y_{i-1}^l\})$. Similarly, if X_i is a lower functional record at time i then $\mathrm{fD}(X_i; \{X_i, Y_{i-1}^u, Y_{i-1}^l\})$ can only tie with $\mathrm{fD}(Y_{i-1}^u; \{X_i, Y_{i-1}^u, Y_{i-1}^l\})$.

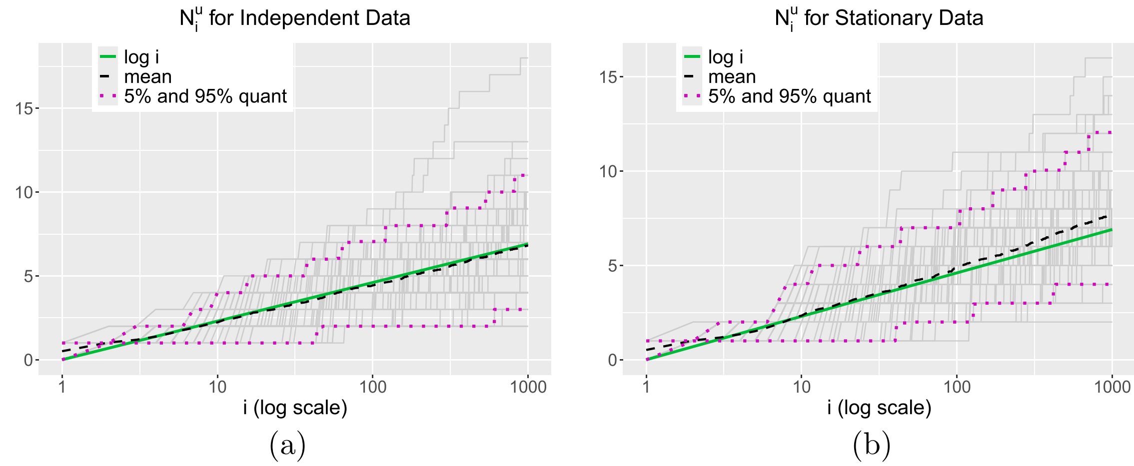

Figure 2. Plots of 100 trajectories of N_i^u by using MBD as the functional depth (results are the same when using ED) with $i = 2, \ldots, 1000$. Each trajectory of N_i^u is obtained from $\{X_i\}_{i=1}^n$ where $n = 1000$, and $\{X_i\}$ is an independent functional sequence (a) and a stationary functional sequence (b). The solid curve represents the log i function, the dashed curve represents the pointwise mean of the trajectories, and the dotted curves represent the pointwise 5% and 95% quantiles of the trajectories.

In the univariate case, it is known that if the time series is an iid sequence or a stationary time series satisfying the Berman condition, then N_i^u grows at rate $\log i$ (Lindgren and Rootzén 1987). On the other hand, if the time series is a random walk process, then the growth rate of N_i^u is $i^{1/2}$ (Sparre Andersen 1954; Feller 1971; Burridge and Guerre 1996). Here, we show similar results for functional records according to Definition 1.

We observe that if $\{X_i, i \geq 1\}$ is an independent and identically distributed sequence of functional random variables, then $P(R_i^u = 1) = 1/i$ for any ranking definition. Indeed, the probability of X_i being a record is the probability of X_i taking a specific place among $\{1, \ldots, i\}$. Then, $N_i^u = O(\log i)$ with probability one. Let $C_h := C_{X_i, X_{i+h}}$ be the covariance operator at lag h of a stationarity functional time series $\{X_i\}$. Let $\| \cdot \|_{\mathcal{HS}}$ denote the Hilbert-Schmidt norm.

Proposition 1. Let $\{X_1, \ldots X_n\}$ be a stationary functional time series such that $\log(h)\|C_h\|_{\mathcal{HS}} \to 0$ as $h \to \infty$. Then,

$$\lim_{n\to\infty} \frac{N_n^u}{\log n} = O(1),$$

with probability one.

Proof. See the supplementary materials. □

The condition on the covariance operator in Proposition 1 is not restrictive for functional time series, and it holds if the functional time series is L^2-m-approximable (Hörmann and Kokoszka 2010). Hörmann and Kokoszka (2010) showed that this approximation is valid for linear and nonlinear functional time series. In particular, the FAR(1) model with a coefficient operator that has a norm less than one is L^2-m-approximable.

As an illustration of the result in Proposition 1, we simulate functional data from two scenarios: independent functional data and stationary functional data. In the first scenario, we simulate $X_i = \varepsilon_i, i = 1, \ldots, n = 1000$ as an independent sequence, where, for each i, ε_i is a Brownian motion in $[0, 1]$. In the second scenario, we simulate stationary functional time series from $X_i(s) = c_1 \int_0^1 \beta(u, s) X_{i-1}(u) \mathrm{d}u + \varepsilon_i(s)$, where $\beta(u, s) = \exp\{-(u^2 + s^2)/2\}$, ε_i is defined as in the first scenario, and c_1 is such that $\left\{\int_0^1 \int_0^1 c_1^2 \beta(u, s)^2 \mathrm{d}u\mathrm{d}s\right\}^{1/2} = 0.5$. Figure 2(a) and (b) show the 100 trajectories of $\{N_i^u;\ i = 2, \ldots, n\}$, using MBD (the results are the same if fD is ED). In each plot, we also present the pointwise mean (dashed curve) and the pointwise 5% and 95% quantiles (dotted curves) of the 100 trajectories. We observe that N_i^u has the same growth rate in all cases, that is, $O(\log i)$.

Now, we state the result for the sequence of random variables N_n^u under a nonstationary functional process.

Proposition 2. Let $\{X_1, \ldots, X_n\}$ be a sequence of functional time series following the model $X_i = X_{i-1} + \varepsilon_i$, with $\{\varepsilon_i\}$ an iid sequence of functional random variables. Also, let ε_0 have a symmetric distribution with zero mean. Then

$$\frac{N_n^u}{\sqrt{n}} \xrightarrow{d} G_1, \tag{4}$$

when $n \to \infty$, where G_1 is a random variable with probability density function $g_1(u) = \frac{1}{\sqrt{\pi}} \exp\left(-u^2/4\right)$ for $u \geq 0$.

Proof. See the supplementary materials. □

In Section 5, we extend Proposition 2 to stationary innovations $\{\varepsilon_i\}$. As an illustration of the result in Proposition 2, we simulate a functional random walk, with Brownian motion in $[0, 1]$, as a functional white noise, and for different sample sizes $n = 100$, and $n = 2000$. We simulate 100 replicates of each case, and then obtain 100 replicates of values of N_n^u (the results are the same for both fDs). Figure 3 shows histograms of $N_n^u/\sqrt{n}$ where the solid blue curve represents the asymptotic distribution from Proposition 2. We observe that the asymptotic distribution provides a better description of the empirical distribution when the sample size increases. However, with a sample size $n = 100$, this approximation is already reasonably good.

5. Application to Functional Unit Root Test

Records have been used in different problems, in particular, to test for a unit root (see Burridge and Guerre 1996; Aparicio,

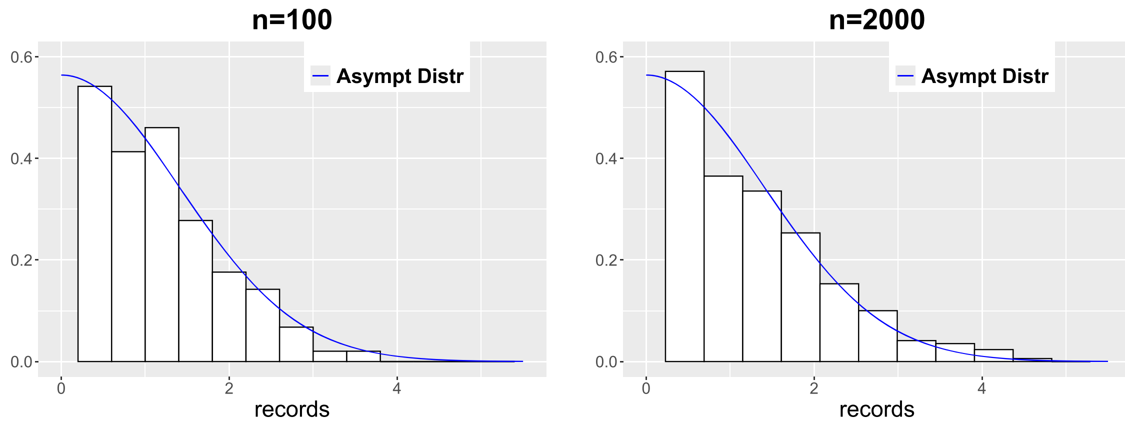

Figure 3. Histogram of values of $N_n^u/\sqrt{n}$ with $n = 100, 2000$, and the asymptotic distribution (solid blue curve) from Proposition 2.

Escribano, and Sipols 2006). In this section, we propose a unit root test for functional time series that uses the normalized counting process $n^{-1/2}N_n := n^{-1/2}(N_n^u + N_n^l)$. One advantage of using records to test for a unit root is that it is a nonparametric test; it is also robust against structural breaks, and it does not involve the estimation of any coefficient operators, which could be a difficult task and, therefore, face computational issues. Moreover, a unit root test based on records will be invariant under monotonic transformations of the data, since functional depths are invariant. Thus, we can use records to test a unit root in a general class of $I(1)$ functional processes.

5.1. Functional Unit Root Test

Loosely speaking, an $I(1)$ functional process has two components: a functional random walk component and a stationarity functional process component. Specifically, we consider the following $I(1)$ functional process

$$X_i(s) = \mu(s) + \sum_{j=1}^{i} \varepsilon_j(s) + \eta_i(s), \tag{5}$$

where $\{\varepsilon_i\}$ is a sequence of iid functional random variables, and $\{\eta_i\}$ is a stationary functional time series. A simple example of an $I(1)$ functional process is the functional random walk $X_i(s) = X_{i-1}(s) + \varepsilon_i(s)$, which can be written as $X_i(s) = X_0(s) + \sum_{j=1}^{i} \varepsilon_j(s)$. Intuitively, the trajectory of the functional random walk component in (5) leads the trajectory of the $I(1)$ functional process. Thus, the number of records for the $I(1)$ functional process is expected to be similar to that of a functional random walk. We formalize this result in the following proposition.

Proposition 3. Let $\{X_i\}$ be an $I(1)$ functional process as defined in (5) with $\{\varepsilon_i\}$ having symmetrical distribution and zero mean. Then, the corresponding normalized random variable $N_n^u/\sqrt{n}$ has the same asymptotic distribution as the corresponding one for a functional random walk in Proposition 2.

Proof. See the supplementary materials. □

Next, we define a record-based (RB) unit root test, which we refer to as RB-fURT in the tables below. We consider the testing of the null hypothesis of an $I(1)$ functional process versus the functional process being stationary. In other terms,

$$H_0 : \{X_i\} \text{ is an } I(1) \text{ functional process} \quad \text{versus}$$
$$H_1 : \{X_i\} \text{ is a stationary process,}$$

where the corresponding innovations $\{\varepsilon_i\}$ are assumed to be symmetrically distributed. For sample size n, the test statistic T_n for the RB-functional unit root test is the number of upper and lower records normalized with $\sqrt{n}$, that is, $T_n = n^{-1/2}N_n = n^{-1/2}(N_n^u + N_n^l)$.

Corollary 1. Let $\{X_i\}_{i=1}^n$ be a realization of a functional time series. We have that

1. under the null hypothesis, $T_n \xrightarrow{d} G_2$, where G_2 is a random variable with probability density function $g_2(u) = \sqrt{\frac{2}{\pi}}\, u^2 \exp(-u^2/2)$, $u \geq 0$, and
2. under the alternative hypothesis, $T_n \xrightarrow{p} 0$.

Proof. See the supplementary materials. □

From Corollary 1, we use the left tail of the asymptotic distribution of the test statistic T_n to test for a functional unit root, that is, given the significance level α, reject H_0 if T_n is smaller than the quantile q_α of order α of $g_2(u)$.

In the following sections, we present a Monte Carlo simulation study to evaluate the performance of the test for a finite sample size. Furthermore, we conduct a comparison of our results with the functional unit root tests that already exist in the literature.

5.2. Simulation Design and Existing Methods

In this section, we study the performance of our proposed unit root test based on functional records under the null and alternative hypothesis.

In the literature, we can find methods to test for a functional unit root. In Kokoszka and Young (2016), a KPSS test was proposed to test unit root for functional time series. The KPSS test assumes that the null hypothesis is $X_i(s) = \mu(s) + i\xi(s) + \eta_i(s)$, where μ is the mean, $i\zeta$ is a "linear" trend, and η_i is a stationary functional time series. The alternative hypothesis is $X_i(s) =$

$\mu(s) + i\xi(s) + \sum_{j=1}^{i} \varepsilon_j(s) + \eta_i(s)$. Then, a proper test statistic is defined (see the corresponding paper for more details). We compare our method with this KPSS test and assume there is no linear trend component, that is, $\xi \equiv 0$. Another method can be found in Chen and Pun (2021a). In that paper, the null hypothesis is defined as $X_i(s) = \mu(s) + \sum_{j=1}^{i} \eta_i(s)$, and the alternative hypothesis is either trend stationary or weakly dependent stationary. Here, we use this method with the weakly dependent stationary as an alternative. In our results, this method is denoted as fURT (see Chen and Pun (2021a) for more details). Another paper that deals with functional unit root is Horváth, Kokoszka, and Rice (2014). However, the test is the same as the KPSS test when $\xi \equiv 0$. Thus, we compare our method with two tests, KPSS and fURT.

We simulate different functional time series, $\{X_i(s)\}_{i=1}^{n}$, at 50 points[2] equispaced on $[0, 1]$ with different sample sizes $n = 200, 300, 500$, and 1000. Each scenario is replicated 100 times. Let $\{\varepsilon_i(s)\}$ be a sequence of iid functional random variables. We consider the following models:

1. $X_i(s) = X_{i-1}(s) + \varepsilon_i(s)$;
2. $X_i(s) = \sum_{j=1}^{i} \varepsilon_j(s) + \eta_i(s)$, where η_i is a stationary FAR(1) process as in Model 4;
3. $X_i(s) = \varepsilon_i(s)$;
4. $X_i(s) = \Psi_1(X_{i-1})(s) + \varepsilon_i(s)$, where $\Psi_1(z) = c_1 \int_0^1 \exp\{(u^2 + s^2)/2\} z(u) du$ and c_1 is such that $\|\Psi_1\|_{\mathcal{HS}} = 0.5$;
5. $X_i(s) = \mu_1(s)\mathbb{1}_{\{i \le k\}} + \mu_2(s)\mathbb{1}_{\{i > k\}} + \eta_i(s)$, where $\eta_i(s)$ is a stationary FAR(1) process as in Model 4, $\mu_1(s) = 0$, $\mu_2(s) = 2$ and $k = n/2$; and
6. $X_i(s) = (\Psi_1 \mathbb{1}_{\{i > k\}} + \Psi_2 \mathbb{1}_{\{i \le k\}})(X_{i-1})(s) + \varepsilon_i(s)$, where Ψ_1 is as in Model 4 and $\Psi_2(z) = c_2 \int_0^1 \exp\{-(u^2 + s^2)/2\} z(u) du$ where c_2 is such that $\|\Psi_2\|_{\mathcal{HS}} = 0.7$, and $k = n/2$.

Models 1 and 2 are functional time series with a unit root component, that is, $\{X_i\}$ is an $I(1)$ functional process, whereas, in Models 3–6, $\{X_i\}$ is not an $I(1)$ functional process. Particularly, in Models 3 and 4, $\{X_i\}$ is stationary. We consider the functional white noise to be the Brownian motion (Bm), $\varepsilon_i(s) = W_i(s), s \in [0, 1]$, the Brownian bridge (Bb), $\varepsilon_i(s) = W_i(s) - sW_i(1), s \in [0, 1]$, and $\varepsilon_i(s)$ as a stochastic Gaussian process ($\mathrm{Gp}(0, \gamma)$) with zero mean and covariance function $\gamma(s, u) = 0.2 \exp\{-0.3|s - u|\}$ in $[0, 1]$.

5.3. Empirical Size and Power of the Test

For each simulation, we compute our test statistic T_n using MBD and ED as the functional depths (the results are the same for both fDs), and then we compare it with the quantile $q_{0.05}$ obtained from the asymptotic distribution in Corollary 1 (our method is denoted as RB-fURT). Similarly, for the KPSS and fURT methods, we compare the corresponding test statistics with the corresponding 0.05 quantiles. Table 1 presents the proportion of rejections when the functional time series is under the null hypothesis of our model. First, we describe the results of our method. We observe that for Model 1 with white noise Bm and depth MBD, the proportion of rejection is 0.023 when $n = 200$, and it increases to 0.038 when $n = 1000$. We observe

Table 1. Empirical size of our method.

	Model 1				Model 2			
n	200	300	500	1000	200	300	500	1000
RB − fURT								
ε_i								
Bm	0.023	0.033	0.017	0.038	0.080	0.060	0.037	0.050
Bb	0.001	0.003	0.002	0.009	0.057	0.040	0.010	0.007
$\mathrm{Gp}(0,\gamma)$	0.020	0.040	0.027	0.025	0.103	0.123	0.070	0.067
fURT								
ε_i								
Bm	0.113	0.113	0.107	0.143	0.547	0.517	0.563	0.527
Bb	0.183	0.180	0.193	0.177	0.723	0.757	0.720	0.720
$\mathrm{Gp}(0,\gamma)$	0.107	0.107	0.087	0.110	0.407	0.393	0.417	0.410
KPSS								
ε_i								
Bm	0.006	0.00	0.000	0.000	0.007	0.000	0.000	0.000
Bb	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$\mathrm{Gp}(0,\gamma)$	0.037	0.017	0.000	0.000	0.037	0.013	0.000	0.000

NOTE: Proportion of rejections under the null hypothesis. For our method RB-fURT, the functional records are obtained using MBD and ED (results are the same for both fDs). We simulate functional time series from Models 1 and 2 with different functional white noises, Brownian motion (Bm), Brownian bridge (Bb), and Gaussian process with zero mean and covariance function γ ($\mathrm{Gp}(0,\gamma)$). The sample sizes considered are $n = 200, 300, 500$, and 1000. Each scenario is replicated 100 times. The nominal level is 5%.

similar results when the white noise is Bb and $\mathrm{Gp}(0, \gamma)$. This suggests a slow convergence rate to the left tail of the asymptotic distribution. For Model 2, the results and conclusion are similar to Model 1, except for the functional white noise Bm; in this case, the proportion of rejection is 0.05 when sample size is $n = 1000$.

Now, we describe the results for the fURT and KPSS methods. We found that the proportion of rejection remains consistent for the fURT method across different sample sizes and white noises in each model. In Model 1, this proportion ranged from 0.1 to 0.19. In Model 2, the results are not good, with a rejection proportion around 0.39 when white noise is $\mathrm{Gp}(0, \gamma)$, 0.52 for Bm, and 0.72 for Bb. Therefore, fURT performs poorly on Model 2. Finally, for the KPSS method, we observe that the proportion of rejection is zero for almost all scenarios. This means that KPSS performs well in both models (zero values are expected since this is a different null hypothesis).

Our next step is to study the power of our proposed test. Table 2 presents the proportion of rejections under the alternative of our method. Note that Model 3 represents a stationary, independent sequence of functional data, whereas Model 4 represents stationary, dependent functional data. Similar to our previous analysis, we begin by describing the results of our method. In Model 3, the proportion of rejections is bigger than 0.93 for small sample sizes, independently of the selection of the white noise ε_i. In Model 4, the proportion of rejections is bigger than 0.96 for sample sizes bigger than $n = 300$. In general, the test shows a high power, even for the smaller sample size, $n = 200$.

Regarding the fURT method, we observe that the rejection rate is one in Models 3 and 4 (which indicates a good performance). However, fURT presented a high size (especially in Model 2). Thus, this rejection rate is not very trustworthy. For the KPSS method, we observe a good performance in Models 3 and 4 (for the KPSS method, the proportion of rejection is expected to be around 0.95).

[2]The results remain consistent if the number of points varies from 10 to 500. See the supplementary materials.

Table 2. Empirical power of our method.

n	Model 3 200	Model 3 300	Model 3 500	Model 3 1000	Model 4 200	Model 4 300	Model 4 500	Model 4 1000
RB-fURT								
ε_i								
Bm	0.97	0.99	1.00	1.00	0.92	0.98	0.99	1.00
Bb	0.93	0.95	0.98	1.00	0.89	0.96	0.99	1.00
Gp(0, γ)	0.95	0.99	1.00	1.00	0.95	1.00	1.00	1.00
fURT								
ε_i								
Bm	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Bb	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Gp(0, γ)	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
KPSS								
ε_i								
Bm	0.96	0.96	0.94	0.95	0.96	0.95	0.95	0.94
Bb	0.93	0.94	0.95	0.97	0.94	0.94	0.93	0.94
Gp(0, γ)	0.96	0.96	0.96	0.94	0.94	0.96	0.95	0.93

NOTE: Proportion of rejections under the alternative hypothesis. For our method RB-fURT, the functional records are obtained using MBD and ED (results are the same for both fDs). Functional time series are simulated from Models 3 and 4 with different functional white noises, Brownian motion (Bm), Brownian bridge (Bb), and Gaussian process with zero mean and covariance function γ (Gp(0, γ)). The sample sizes considered are $n = 200, 300, 500$, and 1000. Each scenario is replicated 100 times.

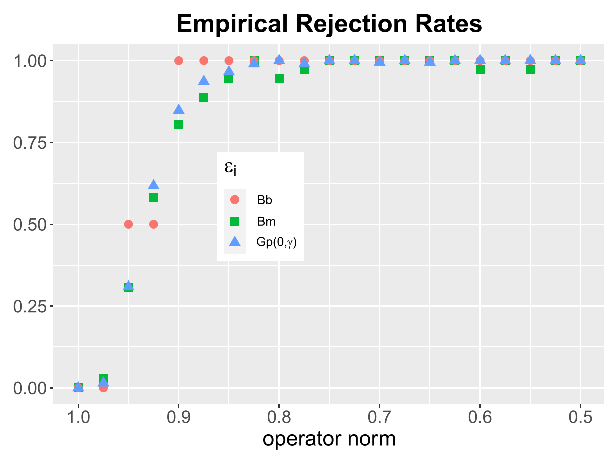

Figure 4. Rejection rate of our method when varying operator norms of the coefficient operator in Model 4 (results are the same for both ED and MBD).

To conclude, from Tables 1 and 2, our RB-functional unit root test shows a good performance in Models 1–4. Similarly, the KPSS test demonstrates good performance in all four models. However, the fURT test has a weak performance in Models 1 and 2.

Additionally, we investigate the power curve of our method for Model 4 by varying the operator norm $\|\Psi_1\|_{\mathcal{HS}} = 0.5, 0.525, \ldots, 0.975, 1$. Figure 4 shows the rejection rate at level $\alpha = 0.05$ with $n = 500$ for each different operator norm. We observe that the test has good power, correctly rejecting the null hypothesis when operator norms are smaller than 0.9.

5.4. Robustness Against Structural Changes

One of the advantages of using functional records in the hypothesis test is the robustness to different nonstationary models. Models 5 and 6 represent unstable time series, with a change in the mean and a change in the coefficient operator, respectively.

Table 3. Proportion of rejections against models with structural changes.

n	Model 5 200	Model 5 300	Model 5 500	Model 5 1000	Model 6 200	Model 6 300	Model 6 500	Model 6 1000
RB-fURT								
ε_i								
Bm	0.49	0.74	0.91	1.00	0.80	0.95	0.98	1.00
Bb	0.45	0.62	0.83	1.00	0.79	0.92	0.97	0.99
Gp(0, γ)	0.65	0.88	0.99	1.00	0.88	0.98	1.00	1.00
fURT								
ε_i								
Bm	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Bb	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Gp(0, γ)	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00
KPSS								
ε_i								
Bm	0.00	0.00	0.00	0.00	0.92	0.95	0.93	0.97
Bb	0.00	0.00	0.00	0.00	0.92	0.95	0.94	0.95
Gp(0, γ)	0.00	0.00	0.00	0.00	0.96	0.93	0.96	0.96

NOTE: For our method RB-fURT, the functional records are obtained using MBD and ED (results are the same for both fDs). Functional time series are simulated from Models 5 and 6 with different functional white noises, Brownian motion (Bm), Brownian bridge (Bb), and Gaussian process with zero mean and covariance function γ (Gp(0, γ)). The sample sizes considered are $n = 200, 300, 500$, and 1000. Each scenario is replicated 100 times.

However, Models 5 and 6 are not $I(1)$ functional processes, so we expect to reject H_0. The counting processes N_i for Models 5 and 6 should grow at the same rate as that in the stationary case: $N_i = O(\log i)$. Table 3 shows the corresponding proportion of rejections for these models.

First, we describe the results of our method. For Model 5, we observe a low proportion of rejections when the sample size is small. In this scenario, for a reasonable power, the test requires a sample size bigger than 300. For Model 6, the results are different. For $n = 200$, we observe that the proportions of rejection of the null hypothesis are bigger than 0.79 for all white noises. Overall, we obtain a proportion of rejections around 0.93 when $n \geq 300$. In general, the RB-functional unit root test is robust against structural changes, although a bigger sample size is needed when changes occur in the mean.

The fURT method seems to be robust in principle as well. The proportion of rejection is one for all cases of structural changes, even for small sample sizes. However, due to the high size of this hypothesis test, one needs to be careful with any conclusion on power.

Regarding the KPSS test, we observe poor performance for Model 5 with zero proportions of rejection. On the other hand, the performance of KPSS is better in Model 6. We obtain a proportion of rejection of around 0.95 for all types of white noise (0.95 is the proportion expected to be observed). Thus, KPSS seems to be robust to changes in the coefficients but not to changes in the mean.

After our simulation study, we conclude that our method outperforms both the fURT and KPSS methods. The fURT test struggles with data obtained from Models 1 and 2, while the KPSS method struggles with data obtained from Model 5. The main benefit of our method is that it is a nonparametric test, so it does not require any parameter estimation. Furthermore, since fD has invariance properties, our method is invariant under monotonic transformations of the data. Thus, our method is expected to be robust against more complex scenarios, including those considered in Models 5 and 6.

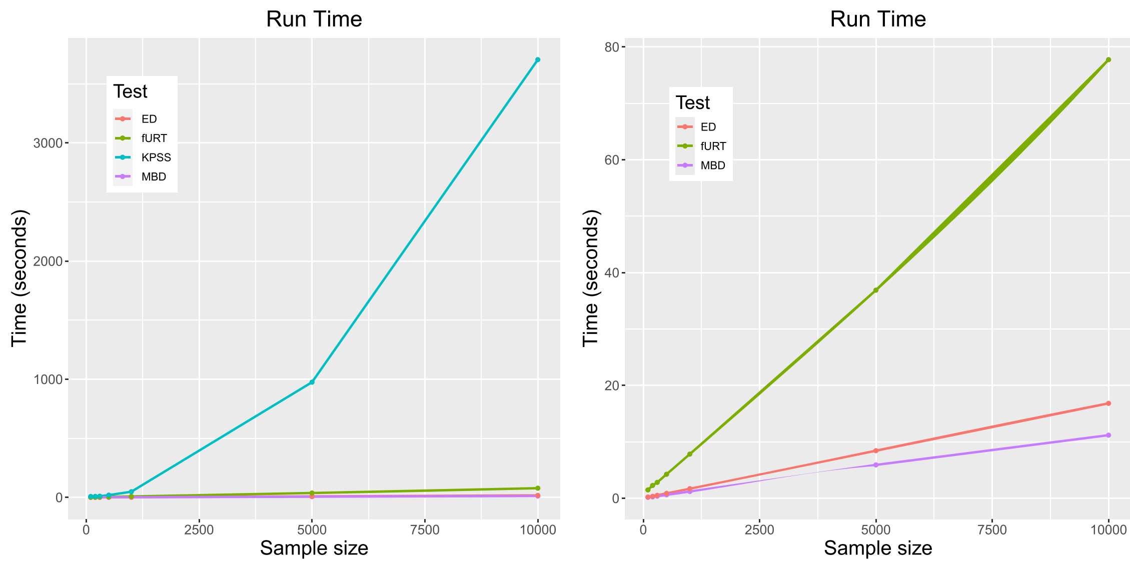

Figure 5. Run time in seconds of different hypothesis testing. Left: all tests. Right: Same as the left figure but excluding the KPSS test. We simulate functional time series with sample size varying from $n = 100$ to $n = 10{,}000$. Each function is evaluated at 100 points equispaced on [0, 1]. ED and MBD represent our proposed unit root test by using functional depths ED and MBD.

5.5. Computational Cost

We end this section by showing the run times of our RB-fURT method and the fURT and KPSS methods. Our proposal is mostly concerned with a definition of functional records and their use in a functional unit root test and as an exploratory analysis tool. Thus, we do not provide an in-depth study of the computational cost.

The run time was measured using a standard laptop with an Intel Core i5 − 1135G7 CPU with 2.40 GHz × 8 Core and 15.3 Gb of RAM. We simulate functional time series, $\{X_i(s)\}_{i=1}^{n}$ using Model 1 with different sample sizes, $n = 100, 200, 300, 500, 1000, 5000, 10{,}000$. Each X_i is evaluated at 100 points equispaced on [0, 1], and then we perform the test. Here, we considered the functional depths MBD and ED to obtain the functional records. Figure 5 presents the run times, where our method is denoted as MBD and ED to emphasize the depth used in the RB-fURT method. We observe that, with sample size $n = 500$, the run times are 2 sec, 2 sec, 4 sec, and 19 sec for MBD, ED, fURT, and KPSS methods, respectively. When the sample size increases to $n = 10{,}000$, the run time of our method with depth MBD is around 10 sec and 20 sec with ED, for fURT is around 1 min, and for KPSS is around 61 min. Here, we see that the computational time of our method is small compared with the competitors.

In this section, we have used the packages *fda* (Ramsay, Graves, and Hooker 2020) and *fdaoutlier* (Ojo, Lillo, and Fernandez Anta 2021) for MBD and ED functional depths, respectively. To conduct the fURT test, we have used the package *STFTS* (Chen and Pun 2021b), and for the KPSS test, we have used an R code provided by the corresponding author.

6. Data Applications

In this section, we apply the different tools described in this article to two different datasets. First, we consider daily curves of the hourly wind speed taken at Yanbu, Saudi Arabia. Our second example involves the annual mortality rates in France (from the R package *demography*, Hyndman et al. 2019), from 1816 to 2006. The functional depths used to obtain the functional records are MBD and ED. The results are the same. Thus, we present the results without specifying the functional depth used.

6.1. Wind Speed in Saudi Arabia

The dataset consists of $n = 755$ daily curves of wind speed at Yanbu, Saudi Arabia, from August 30, 2014 to September 22, 2016. Each point of the curve represents wind speed at 80m $[m/s]$. The study of the behavior of wind speed is important for renewable energy generation. Particularly, by knowing when and how often a record curve of wind speed is observed, we can describe the dynamics of the extreme wind speed curves. An accurate characterization of the extreme daily curves is crucial to predict the efficiencies of wind turbines and energy storage in the presence of an extreme event.

We exclude the first two curves, which are functional records by definition. We found that the functional records for 2014 are: Sept. 2, 5, 26, and Oct. 5, 8, 10. The record curves for 2015 are: Jan. 20, Feb. 2, 15, Apr. 24, and Nov. 22. The record curve for 2016 is Sept. 5. We plot the results in Figure 6. Curves not classified as records are indicated in gray. The lower functional records are indicated by the blue curves, and the upper functional records are indicated by the red curves. We indicate the corresponding year using different line types: 2014-dotted curves, 2015-dashed curves, and 2016-solid curves. We observe that all lower records are in 2014, whereas upper records are in 2015 and 2016. Thus, curves showing the lowest speeds are in Autumn, when the temperature starts to decrease slowly. Most of the upper functional records were observed in Spring and Summer (except the last one observed in September 2016). Summer in Saudi Arabia brings sandstorms driven by Summer South winds. Therefore, it seems reasonable to observe these extreme curves.

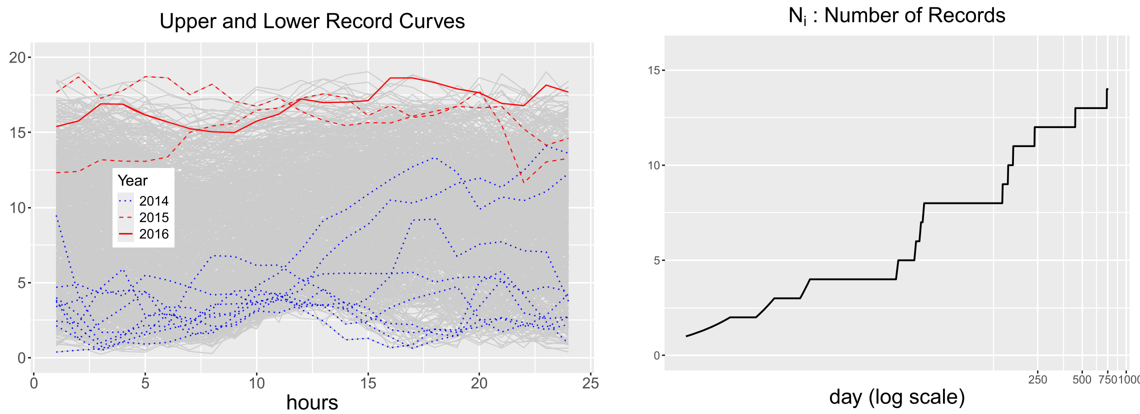

Figure 6. Functional records of daily wind speed at Yanbu, Saudi Arabia, from August 30, 2014 to September 22, 2016. Left: The Functional records. Dotted lines (blue) indicate lower records and solid and dashed lines (red) indicate upper records. Right: Trajectory of the number of functional records $N_i, i = 2, \ldots, 755$.

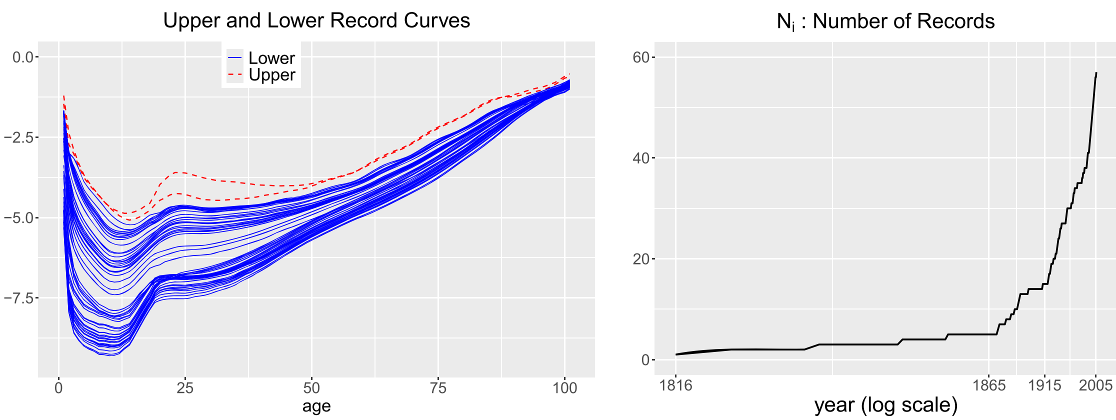

Figure 7. Functional records of log mortality rates in France from 1816 to 2006, for zero to 100 years of age. Left: The functional records. Solid curves indicate lower records and dashed curves indicate upper records. Right: The trajectory of the number of functional records over time, N_i.

We can now infer the class of the underlying functional process. On the right side of Figure 6, we present the trajectory of the corresponding N_i process. We apply our RB-functional unit root test to the wind speed dataset. The test statistic value is $T_n = 0.43$. The corresponding 5% quantile is $q_{0.05} = 0.59$. Thus, the T_n value is smaller than $q_{0.05}$. The corresponding p-value is 0.0209. Thus, we have significant evidence against the stochastic trend and conclude that the functional wind data do not have a unit root component. Therefore, the daily curves of the wind speed can be modeled with a stationary functional time series model.

6.2. Mortality Rates in France

This dataset consists of $n = 191$ curves of annual mortality rates in France, from 1816 to 2006, for zero to 110-years old individuals. However, we consider only up to 100 years of age in order to avoid highly noisy measurements. Each point of the curve $X_i(s)$ represents the total mortality rate, in year i, at age s. Our interest is to study the behavior of the rates, over the years, taking into account all ages. By studying records, we analyze whether the new functional records over the years correspond to the natural randomness of the process, or if they indicate a decreasing trend. The data have been analyzed before by Hyndman and Ullah (2007) using a functional approach. They proposed to forecast the age-specific mortality rate by modeling the coefficients obtained by projecting the functional data to the corresponding robust functional principal components. They fitted an ARIMA model to the coefficients, but they did not report the estimated parameters. Evidence of a univariate unit root can be found if we fit the ARIMA model to the first coefficients, for the first eigenfunction. We, therefore, investigate if there is evidence of a functional unit root. In our analysis, we use the smoothed curves, as described in Hyndman and Ullah (2007).

Again, we exclude the first two curves, which are functional records by definition. We find that the years classified as records are: 1821, 1832, 1845, 1871, 1872, 1877, 1881, 1884, 1887, 1888, 1889, 1897, 1913, 1920, 1921, 1923, 1924, 1927, 1930, 1932–1934, 1936, 1937, 1939, 1946–1948, 1955, 1958, 1959, 1961, 1966, 1975, 1977, 1980, 1985–1987, 1990–2004, and 2006. That is 55 functional records in total.

Figure 7 shows the functional records. We indicate the upper and lower records with different line types: upper functional records with a red dashed curve, and lower functional records with a blue solid curve. We observe only two upper records corresponding to the years 1832 and 1871. The rest of the records correspond to lower functional records. In particular, we observe that, after the last functional upper records in 1871, a new functional record represents a lower mortality rate for almost all ages. This suggests the presence of a functional trend.

Finally, we apply our RB-functional unit root test to the dataset. On the right side of Figure 7, we show the trajectory of the corresponding N_i process. The test statistic value is $T_n = 4.1$. The corresponding 5% quantile from the asymptotic distribution under the null hypothesis is $q_{0.05} = 0.59$. Therefore, we do not have any evidence against the $I(1)$ functional process. Thus, to model this dataset, we must consider the existence of both a stochastic trend and a functional deterministic trend. This is consistent with the findings by Hyndman and Ullah (2007) that take into consideration the ARIMA models for the basis coefficients. However, our approach is more general, as we do not consider any specific model.

7. Discussion

In this article, we provided some statistical tools for functional time series analysis and visualization. These tools are based on a record definition for functional data. We used a depth notion to say when a function crosses another function. We showed that the counting process corresponding to the number of functional records grows at rate $\log n$, for stationary functional time series, and that it grows at rate $n^{1/2}$, for nonstationary functional time series. A simulation study showed that the asymptotic distribution of the number of records has a good approximation when the functional data are a functional random walk, even for small sample sizes.

As a particular application of the extended functional record, we proposed a functional unit root test for $I(1)$ functional processes. Using a Monte Carlo simulation study, we showed that the test performance is good for stationary and nonstationary functional processes. Our test is robust against structural changes for a moderate sample size. The unit root test based on functional records does not assume any model. In the data application, we found that the definition of functional records provides relevant and consistent information about extreme curves. In addition, it allows us to infer the underlying process.

The functional record introduced in this article equates to the maximum and minimum values when observations are real numbers. However, in certain cases, our functional record definition may not be effective in identifying extreme curves. For instance, a functional data could be an extreme curve, but if it does not exceed the previous "maximum" ("minimum") curve, it will not be defined as a record curve. In scenarios like temperature curves, it may be more useful to detect abnormal curves rather than record curves. One potential solution to the problem of detecting abnormal curves is to consider ordering the functional data and then use the quantile concept. Furthermore, one could define a threshold curve and study the curves that exceed this threshold curve. These latter ideas are commonly used in extreme value theory and will be further explored in future works.

The link *https://github.com/I-MH/Functional-Records* corresponds to the R code in GitHub repositories to estimate the functional records.

Supplementary Materials

This article has supplementary material that contains:

Appendix.pdf: A document containing proofs of the Propositions and the Corollary. Also, it contains results of simulation studies for sparse and dense functional data.

R code: A set of R scripts, including the R code to compute the record times and reproduce some data analysis figures.

README.txt: A text file describing all supplementary materials, including all R scripts.

Acknowledgments

This research was supported by the King Abdullah University of Science and Technology (KAUST). The authors thank the King Abdullah City for Atomic and Renewable Energy (K.A. CARE) for providing the wind speed observational data. Also, the authors thank the associate editor and the referees for their thoughtful and constructive comments and suggestions.

Disclosure Statement

The authors report there are no competing interests to declare.

Data Availability Statement

The wind speed data used in this article are available upon request to the authors. The second data used in this article are available from the *demography* R package *https://cran.r-project.org/src/contrib/demography_1.22.tar.gz* (Hyndman et al. 2019).

ORCID

Israel Martínez-Hernández http://orcid.org/0000-0002-4122-2529

References

Ahsanullah, M., and Nevzorov, V. B. (2015), *Records via Probability Theory*, Volume 6 of Atlantis Studies in Probability and Statistics. Paris: Atlantis Press. [72,74]

Aparicio, F., Escribano, A., and Sipols, A. E. (2006), "Range Unit-Root (RUR) Tests: Robust Against Nonlinearities, Error Distributions, Structural Breaks and Outliers," *Journal of Time Series Analysis*, 27, 545–576. [77]

Aue, A., Norinho, D. D., and Hörmann, S. (2015), "On the Prediction of Stationary Functional Time Series," *Journal of the American Statistical Association*, 110, 378–392. [73]

Ballerini, R., and Resnick, S. I. (1987), "Records in the Presence of a Linear Trend," *Advances in Applied Probability*, 19, 801–828. [72]

Bosq, D. (2000), *Linear Processes in Function Spaces*, Volume 149 of Lecture Notes in Statistics. New York: Springer-Verlag. Theory and applications. [73,74]

Burridge, P., and Guerre, E. (1996), "The Limit Distribution of Level Crossings of a Random Walk, and a Simple Unit Root Test," *Econometric Theory*, 12, 705–723. [72,76]

Chakraborty, A., and Chaudhuri, P. (2014), "The Spatial Distribution in Infinite Dimensional Spaces and Related Quantiles and Depths," *The Annals of Statistics*, 42, 1203–1231. [73]

Chen, Y., and Pun, C. S. (2021a), "Functional Unit Root Test," available at SSRN: *https://ssrn.com/abstract=3761262\/*. [78]

——— (2021b), *STFTS: Statistical Tests for Functional Time Series*, R package version 0.1.0. [80]

Dai, W., and Genton, M. G. (2018), "Multivariate Functional Data Visualization and Outlier Detection," *Journal of Computational and Graphical Statistics*, 27, 923–934. [72]

Dombry, C., and Zott, M. (2018), "Multivariate Records and Hitting Scenarios," *Extremes*, 21, 343–361. [72]

Falk, M., Khorrami Chokami, A., and Padoan, S. A. (2018), "On Multivariate Records from Random Vectors with Independent Components," *Journal of Applied Probability*, 55, 43–53. [72]

Feller, W. (1971), *An Introduction to Probability Theory and its Applications. Vol. II* (2nd ed.), New York-London-Sydney: Wiley. [72,76]

Fraiman, R., and Muniz, G. (2001), "Trimmed Means for Functional Data," *TEST*, 10, 419–440. [73]

Gijbels, I., and Nagy, S. (2017), "On a General Definition of Depth for Functional Data," *Statistical Science*, 32, 630–639. [73]

Gnedin, A. V. (1998), "Records from a Multivariate Normal Sample," *Statistics & Probability Letters*, 39, 11–15. [72]

Goldie, C. M., and Resnick, S. (1989), "Records in a Partially Ordered Set," *The Annals of Probability*, 17, 678–699. [72]

Goldie, C. M., and Resnick, S. I. (1995), "Many Multivariate Records," *Stochastic Processes and their Applications*, 59, 185–216. [72]

Hörmann, S., and Kokoszka, P. (2010), "Weakly Dependent Functional Data," *The Annals of Statistics*, 38, 1845–1884. [76]

Horváth, L., Hušková, M., and Kokoszka, P. (2010), "Testing the Stability of the Functional Autoregressive Process," *Journal of Multivariate Analysis*, 101, 352–367. [73]

Horváth, L., Kokoszka, P., and Rice, G. (2014), "Testing Stationarity of Functional Time Series," *Journal of Econometrics*, 179, 66–82. [78]

Huang, H., and Sun, Y. (2019), "A Decomposition of Total Variation Depth for Understanding Functional Outliers," *Technometrics*, 61, 445–458. [73]

Hyndman, R. J., and Shang, H. L. (2010), "Rainbow Plots, Bagplots, and Boxplots for Functional Data," *Journal of Computational and Graphical Statistics*, 19, 29–45. [72]

Hyndman, R. J., and Ullah, M. S. (2007), "Robust Forecasting of Mortality and Fertility Rates: A Functional Data Approach," *Computational Statistics & Data Analysis*, 51, 4942–4956. [81,82]

Hyndman, R. J., with contributions from Heather Booth, Tickle, L., and Maindonald, J. (2019), *demography: Forecasting Mortality, Fertility, Migration and Population Data*. R package version 1.22. [80,82]

Kokoszka, P., and Reimherr, M. (2013), "Determining the Order of the Functional Autoregressive Model," *Journal of Time Series Analysis*, 34, 116–129. [73]

Kokoszka, P., and Young, G. (2016), "KPSS Test for Functional Time Series," *Statistics. A Journal of Theoretical and Applied Statistics*, 50, 957–973. [77]

Lindgren, G., and Rootzén, H. (1987), "Extreme Values: Theory and Technical Applications," *Scandinavian Journal of Statistics*, 14, 241–279. [72,76]

López-Pintado, S., and Romo, J. (2009), "On the Concept of Depth for Functional Data," *Journal of the American Statistical Association*, 104, 718–734. [73,74]

——— (2011), "A Half-Region Depth for Functional Data," *Computational Statistics & Data Analysis*, 55, 1679–1695. [73]

Martínez-Hernández, I., Genton, M. G., and González-Farías, G. (2019), "Robust Depth-based Estimation of the Functional Autoregressive Model," *Computational Statistics & Data Analysis*, 131, 66–79. [73]

Narisetty, N. N., and Nair, V. N. (2016), "Extremal Depth for Functional Data and Applications," *Journal of the American Statistical Association*, 111, 1705–1714. [73,74]

Nieto-Reyes, A., and Battey, H. (2016), "A Topologically Valid Definition of Depth for Functional Data," *Statistical Science*, 31, 61–79. [73]

Ojo, O. T., Lillo, R. E., and Fernandez Anta, A. (2021), *fdaoutlier: Outlier Detection Tools for Functional Data Analysis*. R package version 0.2.0. [80]

Ramsay, J. O., Graves, S., and Hooker, G. (2020), *fda: Functional Data Analysis*. R package version 5.1.4. [80]

Ramsay, J. O., and Silverman, B. W. (2005), *Functional Data Analysis* (2nd ed.), Springer Series in Statistics, New York: Springer. [72,74]

Rousseeuw, P. J., and Hubert, M. (1999), "Regression Depth," *Journal of the American Statistical Association*, 94, 388–402. [73]

Sguera, C., Galeano, P., and Lillo, R. (2014), "Spatial Depth-based Classification for Functional Data," *TEST*, 23, 725–750. [73]

Sparre Andersen, E. (1954), "On the Fluctuations of Sums of Random Variables. II," *Mathematica Scandinavica*, 2, 195–223. [72,76]

Sun, Y., and Genton, M. G. (2011), "Functional Boxplots," *Journal of Computational and Graphical Statistics*, 20, 316–334. [72,73]

Wergen, G., Majumdar, S. N., and Schehr, G. (2012), "Record Statistics for Multiple Random Walks," *Physical Review E*, 86, 011119. [72]